ME451
Kinematics and Dynamics of Machine Systems

Introduction to Dynamics
November 13, 2014

Quote of the day: “I am the greatest, I said that even before I knew I was.”
-- Muhammad Ali
Before we get started…

- **Last time**
  - Wrapped up the Newton-Euler equations of motion for a rigid body
  - Started properties of the Centroid and Mass Moment of Inertia, Inertial Properties of Composite Bodies

- **Today**
  - Finish discussion properties of the Centroid and Mass Moment of Inertia, Inertial Properties of Composite Bodies
  - Enlarging the family of forces that can show up in the equations of motion

- **Project 1 – Due date: Nov 18 at 11:59 PM**
  - Requires you to use simEngine2D in conjunction with excavator example discussed in class
  - Not trivial, requires some thinking

- **New assignment posted online**
  - Has pen-and-paper, ADAMS, and MATLAB components
  - Due in one week, Nov. 20.

- **Final Project proposal due on Nov 18 @ 11:59 PM - post your proposal on the Forum**
  - A discussion thread was started on this topic
  - Dan to provide feedback
Take Away Slide: Newton-Euler EOM

- Here’s what we derived last time: the EOM for a centroidal LRF (CLRF)

\[
\begin{bmatrix}
    m\ddot{r} - F \\
    J'\ddot{\phi} - n
\end{bmatrix} = 0 \quad \Leftrightarrow \quad m\ddot{r} = F \\
J'\ddot{\phi} = n
\]

- Got these second order differential equations starting from “first principles”
  - Newton’s laws for a particle
  - The rigid body assumption

- They tell us what the acceleration of the CLRF slapped on the body looks like
  - Recall that this is what we were after: figuring out what the acceleration is
    - Integrate once to get velocity
    - Integrate once again to get positions
    - (easier said than done)
Mass Moment of Inertia (MMI)
Parallel Axis Theorem

- Also called Polar Moment of Inertia (PMI)
- The MMI with respect to some LRF $O'x'y'$ is by definition the following integral

$$J' = \int_{m} (s'^P)^T s'^P \, dm(P)$$

- Question: Given the value $J'$ calculated with respect to the centroidal frame, what is the value of this integral with respect to the LRF $O''x''y''$?

- Parallel Axis Theorem (Steiner’s Theorem)

$$J'' = \int_{m} (s''P)^T s''P \, dm(P) = J' + m \left( \rho''^T \rho'' \right)$$

Jakob Steiner (1796–1863)
Inertial Properties of Composite Bodies

- Masses, centroid locations, and MMI for rigid bodies with constant density and of simple shapes can be easily calculated.

- Question: how do we calculate these quantities for bodies made up of rigidly attached subcomponents?
  
  - Step 1: Calculate the total body mass \( m = \sum_{i=1}^{k} m_i \)
  
  - Step 2: Compute the centroid location of the composite body \( \rho'' = \frac{1}{m} \sum_{i=1}^{k} m_i \rho_i'' \)
  
  - Step 3: For each subcomponent, apply the parallel axis theorem to include the MMI of that subcomponent with respect to the newly computed centroid, to obtain the MMI of the composite body

\[
J^* = \sum_{i=1}^{k} \left( J_i' + m_i \rho_i^* T \rho_i^* \right)
\]

\( \rho_i^* \) represents the vector from the composite body CM to body \( i \) CM

- Note: if holes are present in the composite body, it is ok to add and subtract material.
Roadmap: Check Progress

What have we accomplished so far?
- Derived the variational and differential EOM for a single rigid body
  - These equations assume their simplest form in a centroidal RF
  - Properties of the mass moment of inertia, figuring out center of mass, etc.

What’s left at this point?
- Define a general methodology for including external forces, concentrated at a given point $P$ on the body
  - Virtual work and generalized forces
- Elaborate on the nature of these concentrated forces. These can be:
  - Models of common force elements (TSDA and RSDA)
  - Reaction (constraint) forces, modeling the interaction with other bodies
- Derive EOM for systems of constrained bodies
6.2

Virtual Work and Generalized Force
Including Concentrated Forces (1/3)

Setup:
- A single rigid body
- Absolute (Cartesian) generalized coordinates using a centroidal frame
- A concentrated force $\mathbf{F}$ acts on the body at point $P$, located by $s''^P$

Question:
- How do we include the force $\mathbf{F}$ in the EOM?

Solution:
- A general methodology is to use D’Alembert’s Principle
- The key steps to derive the generalized force produced by $\mathbf{F}$ are
  - Write down the virtual work produced by $\mathbf{F}$
  - Add this virtual work to the balance of virtual work that shows up in ’Alembert’s principle
Including Concentrated Forces (2/3)

- Rearrange the variational EOM as:

\[
\delta r^T (F - m\ddot{r}) + \delta \phi \left( n - J'\ddot{\phi} \right) = 0
\]

\[
\uparrow
\]

\[
(\delta r^T F + \delta \phi n) + \left( \delta r^T (-m\ddot{r}) + \delta \phi (-J'\ddot{\phi}) \right) = 0
\]

and read this as “the virtual work of the applied (external) forces and the inertial forces is zero for any virtual variations \((\delta r, \delta \phi)\) of the generalized coordinates” (D’Alembert’s Principle)

- A more compact and convenient form uses matrix-vector notation

\[
\begin{bmatrix}
\delta r^T \\
\delta \phi \\
\delta q^T
\end{bmatrix}
\begin{bmatrix}
F \\
n
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
m & 0 & 0 \\
0 & m & 0 \\
0 & 0 & J'
\end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
\ddot{r} \\
\ddot{\phi} \\
\ddot{q}
\end{bmatrix}
= 0
\]

called Generalized Force

\[
\delta q^T (Q - M\ddot{q}) = 0
\]
Including Concentrated Forces (3/3)

\[ \delta q^T (Q - M\ddot{q}) = 0 \]

- **Nomenclature:**
  - \( \ddot{q} \): generalized accelerations
  - \( \delta q \): generalized virtual displacements
  - \( M \): generalized mass matrix
  - \( Q \): generalized forces

- **Recipe for including a concentrated force in the EOM:**
  - Write the virtual work of the given force or torque
  - Express this virtual work in terms of the generalized virtual displacements
  - Identify the generalized force \( Q \) (gather the terms that multiply \( \delta r^T \) and \( \delta \phi \))
  - Include the generalized force in the matrix form of the variational EOM

\[
F \xrightarrow[\text{leads to}]{\delta W = \cdots = \delta r^T Q_1 + \delta \phi Q_2} Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}
\]

(understanding this process will come in handy in your assignment)
Where the Rubber Hits the Road: Q Produced by a Point Force (1/2)

Consider a point force $\mathbf{F}^P$ acting on the body at point $P$ located, with respect to the centroidal LRF, by the vector $\mathbf{s}^P$.

- Write the virtual work done by $\mathbf{F}^P$ over a virtual displacement $\delta \mathbf{r}^P$ of point $P$:
  \[ \delta W = (\delta \mathbf{r}^P)^T \mathbf{F}^P \]

- Use $\mathbf{r}^P = \mathbf{r} + A \mathbf{s}'^P \implies \delta \mathbf{r}^P = \delta \mathbf{r} + \phi A \mathbf{s}'^P$

- Express this virtual work in terms of $\delta \mathbf{q}$:
  \[ \delta W = (\delta \mathbf{r} + \phi A \mathbf{s}'^P)^T \mathbf{F}^P \]
  \[ = \delta \mathbf{r}^T \mathbf{F}^P + \phi \mathbf{s}'^P B^T \mathbf{F}^P \]

- Identify the generalized force $\mathbf{Q}$:
  \[ \mathbf{Q} = \begin{bmatrix} \mathbf{F}^P \\ (\mathbf{s}'^P)^T B^T \mathbf{F}^P \end{bmatrix} = \left[ (\mathbf{B} \mathbf{s}'^P)^T \right] \mathbf{F}^P \]
Where the Rubber Hits the Road: Q Produced by a Point Force (1/2)

What if the point force is better expressed in the LRF’?

Use $F^P = AF'P$ to obtain:

$$Q = \left[ (s'P)^T B^T F^P \right] = \left[ (s'P)^T B^T AF'P \right] = \left[ (s'P)^T (AR)^T AF'P \right] = \left[ (s'P)^T R^T F'P \right]$$

$$Q = \left[ \frac{A}{(Rs'P)^T} \right] F'P$$

**Important:** A concentrated force applied at a point away from the center of mass (that is $s'P \neq 0$) also induces a torque, that is a component in the generalized force that affects the rotational degree of freedom of the body:

$$Q_2 = (s'P)^T B^T F^P = \left( Rs'P \right)^T F'P$$  (see bottom of slide 10 for meaning of $Q_2$)
Including a Torque

What is the generalized force induced by applying a torque $n$ to the body?

Note: no need to specify the point of application for a torque (unlike a force, see previous two slides)

Writing the virtual work done by the torque $n$ and expressing it in terms of the generalized virtual displacements gives:

$$\delta W = \delta \phi \ n$$

and therefore

$$Q = \begin{bmatrix} 0 \\ n \end{bmatrix}$$
Tractor Model
[Example 6.1.1]

- Derive EOM under the following assumptions:
  - Traction (driving) force $T_r$ generated at rear wheels
  - Small angle assumption (on the pitch angle $\phi_1$)
  - Tire forces depend linearly on tire deflection (in reality both tire and terrain deform):

\[
F_t = f(d_t) = \begin{cases} 
kd_t, & d_t \geq 0 \\
0, & d_t < 0 
\end{cases}
\]