Quote of the day: “Computer science education cannot make anybody an expert programmer any more than studying brushes and pigment can make somebody an expert painter.”

-- Eric Raymond
Before we get started...

- Last time
  - Wrapped up wrecker boom example
  - simEngine2D-related MATLAB implementation issues
    - Focus on assembling Phi, Nu, Gamma, and Jac

- Today
  - Virtual displacements and virtual work (this is very esoteric)
  - Start working towards deriving the equations of motion of a rigid body

- Midterm exam: 11/04/2014 (during regular class hours, same room)
  - Open everything
  - Midterm Review: on 11/03/2014
    - Starts at 7:15 pm, runs as long as needed
    - Room TBA

- Project 1 made available online – Due date: Nov 11 at 11:59 PM
  - Requires you to use simEngine2D in conjunction with excavator example discussed in class

- No new homework assigned this week
Final Project Related...

- Proposal for Final Project due in less than three weeks
  - Due 11/18/2014 at 11:59 PM

- Post your Final Project proposal online to get feedback and the go-ahead from me

- You can work on teams of up to two students

- You can choose one of several alternatives:
  - A project that uses your simEngine2D to some useful end
  - A project that uses ADAMS to some useful end
  - A project in which you formalized a general purpose modeling language (for the ADM/ACF files), a parser, data structures and demonstrate your contribution in conjunction with your simEngine2D
  - Other ideas that make sense, are feasible, and the scope of work is sufficient
There will be an automatic A for the student who fulfills the following requirements

- Has the most versatile; i.e., feature-rich, simEngine2D engine
- All the features claimed to be supported by his/her simEngine2D have been validated using ADAMS results to confirm correctness of behavior
- Has got at least a 60% composite grade for the rest of the class
  - Composite grade averages scores from projects, assignments, midterms, etc.

Since it is plausible that a team of two people has worked on simEngine2D both students will get an automatic A grade
The final exam will take place as follows

Part 1: You will be given an ADM/ACF pair of input files for a simple mechanism and asked to use your simEngine2D to run an analysis and generate a couple of plots that show the time evolution of a component of the mechanism

Part 2: You will be given a mechanism and will be asked to

- Provide an ADM/ACF file that models the mechanism
- Use your simEngine2D to determine the time evolution of the mechanism
- Provide plots that show the time evolution of some quantities of interest
  - Example: Position of the center of mass of body 2, reaction force in the revolute joint between body 2 and ground, etc.

Part 3: Zip your simEngine2D in a file, along with the ADM/ACF files and the png images of the plots you obtained and email to negrut@engr.wisc.edu

There will be no pen and paper component for the final exam
Purpose of Chapter 6

- At the end of this chapter you should understand what “dynamics” means and how you should go about carrying out a dynamics analysis.

- We’ll learn a couple of things, probably the most important being:
  - How to formulate and then solve the equations that govern the time evolution of a system of bodies in planar motion
    - These equations are differential equations and they are called equations of motion
    - As many bodies as you wish, connected by any joints we’ve learned about…
  - How to compute the reaction forces in any joint connecting any two bodies in the mechanism
  - How to develop MATLAB code that allows one to carry dynamics analysis of 2D mechanisms
Kinematics vs. Dynamics

- **Kinematics**
  - We include as many actuators as kinematic degrees of freedom
  - We end up with $\text{NDOF} = 0$
    - That is, we have as many constraints as generalized coordinates
  - We find the (generalized) positions, velocities, and accelerations by solving algebraic problems (both nonlinear and linear)
  - We do not care about forces, only that certain motions are imposed on the mechanism. We do not care about body shape nor inertia properties

- **Dynamics**
  - The time evolution of the system is dictated by the applied external forces and motions
  - We have that $\text{NDOF} \geq 0$ yet true dynamics occurs at $\text{NDOF} > 0$
  - The governing equations are differential or differential-algebraic equations
  - Forces acting on bodies and inertia properties of bodies in mechanism are very relevant
Some clarifications

- Dynamics **key** question: how can I get the acceleration of each body of the mechanism?
  - Why is acceleration so relevant? If you know the acceleration you can integrate it twice to get velocity and position information for each body.
  - How is the acceleration of a body “i” measured in the first place?
    - You attach a reference frame on body “i” and measure the acceleration of the body reference frame with respect to the global reference frame:
      \[
      \ddot{q}_i = \begin{bmatrix} \ddot{x}_i \\ \ddot{y}_i \\ \ddot{\phi}_i \end{bmatrix}
      \]

- The answer to the **key** question: To get the acceleration of each body, you first need to formulate the equations of motion
  - Remember **F=ma** from ME240?
    - Actually, the way we state this is **ma=F**, which is the “equation of motion”, that is, what we are after here
Equations of motion of ONE planar RIGID body

- **Framework:**
  - We are dealing with **rigid** bodies
  - For a couple of lectures we’ll consider only one body
    - We’ll extend to more bodies in two weeks…

- **What are we after?**
  - Proving that for one body with a **reference frame attached at its center of mass location** the equations of motion are:

\[
\begin{align*}
m\ddot{\mathbf{r}} - \mathbf{F} &= 0 \quad \text{Equations of Motion governing translation} \\
J'\ddot{\phi} - n &= 0 \quad \text{Equation of Motion governing rotation}
\end{align*}
\]

- \( \mathbf{r} \) is the position of the body local reference frame
- \( \phi \) is the orientation of the body local reference frame
Equations of Motion (EOM)

Some clarifications re previous slide...

- Centroidal reference frame of a body
  - A reference frame located right at the center of mass of that body
  - How is this special? It’s special since a certain integral vanishes...

\[
\int_{m}^{s'P} dm(P) = 0
\]

- What is \( J' \)?
  - Mass moment of inertia

\[
J' = \int_{m}^{[s'P]^T s'P} dm(P)
\]

NOTE: Textbook uses misleading notation

\[
s'^{PT} \Leftrightarrow [s'P]^T
\]
Roadmap, Dynamics Component of ME451
[pretty much from now to end of semester]

- Discuss concept of virtual work and the concept of virtual displacement
  - Principle of virtual work and D’Alembert’s principle used later on
  - Partial derivatives haunting us again (when discussing virtual displacements)

- Write the **differential EOM** for a single rigid body
  - Newton-Euler equations
  - Consider the special case of bodies w/ centroidal reference frame
    - Centroid, mass moment of inertia, parallel axis [Steiner’s] theorem

- Derive the **variational EOM** for constrained planar systems
  - Virtual work and generalized forces

- Formulate and solve the mixed **differential-algebraic EOM** for constrained systems
  - Lagrange multiplier theorem
  - Newmark formula for discretizing differential equations
    - Transforms differential equations into algebraic equations
Two Important Principles

- **Principle of Virtual Work**
  - Applies to a collection of particles
  - States that a configuration is an equilibrium configuration if and only if the virtual work of the forces acting on the collection of particles is zero
    \[ \sum_i \delta \mathbf{r}_i^T \cdot \mathbf{F}_i = 0 \]

- **D’Alembert’s Principle**
  - For a collection of particles experiencing accelerated motion you can still fall back on the Principle of Virtual Work when you also include in the set of forces acting on each particle its inertia force
    \[ \sum_i \delta \mathbf{r}_i^T \cdot (\mathbf{F}_i - m_i \ddot{\mathbf{r}}_i) = 0 \]

  - **NOTE:** we are talking here about a collection of particles
    - Consequently, we’ll have to regard each rigid body as a collection of particles that are rigidly connected to each other and that together make up the body
Remark on the Principle of Virtual Work
- The power of this method stems from the fact that it excludes from the analysis forces that do no work during a virtual displacement
  - In particular constraint forces that are due to kinematic constraint do no work (under certain circumstances)

Remark by Lagrange on D’Alembert’s Principle
- “D’Alembert had reduced dynamics to statics by means of his principle”

NOTE: We will need to come up with expression of the virtual work $\delta W$ produced by a force that moves through a virtual displacements of a point P, that is $\delta r^p$
Virtual Work and Virtual Displacement

- Imagine that force $\mathbf{F}^P$ acts on a rigid body at point $P$. The virtual work done by this force is

$$\delta W^{FP} = [\delta \mathbf{r}^P]^T \cdot \mathbf{F}^P$$

- Here $\delta \mathbf{r}^P$ represents a virtual displacement of the point $P$ in response to a virtual translation and a virtual rotation experienced by the body to which $P$ is attached.
Calculus of Variations

The change $\delta u$ of a function $u(q, t)$ due to a **small** change (or variation) in its argument $q \rightarrow q + \delta q$

- Framework: assume you have a vector function that depends on $q$. Assume that the value of $q$ changes to $q + \delta q$. What is the variation in the function that depends on $q$ due to the said change?

- Specifically, assume the vector function of interest is $u$, and $u$ depends on $q$ and possibly time $t$:

  $$u = u(q, t)$$

- I am interested at a fixed time $t$ in the $\delta u$ below given $q$, $\delta q$, and the expression of $u(q)$:

  $$q \rightarrow u(q, t) \quad q + \delta q \rightarrow u(q + \delta q, t) = u(q, t) + \delta u$$

  $$\delta u = ?$$
Calculus of Variations
[Cntd.]

\[ \delta u = ? \]

- Using a Taylor series expansion argument, one can show that

\[ \delta u = u_q \delta q \]

- NOTE: Pause and reflect on why the sensitivity is bound to show up in how a small change in \( q \) leads to a change in \( u \).

- The function \( u \) can be a scalar function, vector function, or a matrix function.

- It easy to see that given two functions \( f(q, t) \) and \( g(q, t) \),

\[
\delta(f+g) = \delta f + \delta g = f_q \delta q + g_q \delta q \quad \text{and} \quad \delta(f-g) = \delta f - \delta g = f_q \delta q - g_q \delta q
\]
Calculus of Variations
[Cntd.]

One can develop a calculus of variations that resembles extremely closely the differential calculus. The symbol used is changed from $d$ to $\delta$ and time $t$ is always held fixed and plays no role.

For example, if $a(q, t)$ and $b(q, t)$ are two functions of $q$ and $t$:

\[
\begin{align*}
\delta(a + b) &= \delta a + \delta b \\
\delta(\alpha a) &= \delta \alpha a + \alpha \delta a \\
\alpha &= \text{const} \quad \Rightarrow \quad \delta(\alpha a) = \alpha \delta a \\
\delta(a^T b) &= a^T \delta b + b^T \delta a \\
a^T a &= \text{const} \quad \Rightarrow \quad a^T \delta a = 0
\end{align*}
\]
Calculus of Variations
[Cntd.]

\[ u = u(q) \quad \Rightarrow \quad \delta q \xrightarrow{\text{Leads to}} \delta u \]

\[ u(q) = g^T h \quad \Rightarrow \quad \delta u = (g^T h_q + h^T g_q) \delta q \]
\[ u(q) = Bq \quad \Rightarrow \quad \delta u = B \delta q \]
\[ u(q) = q^T B p \quad \Rightarrow \quad \delta u = p^T B^T \delta q \]
\[ u(q) = g^T B h \quad \Rightarrow \quad \delta u = (g^T B h_q + h^T B^T g_q) \delta q \]

Assumptions:

- \( g = g(q) , \ h = h(q) \)
- \( B \) is a constant matrix
- \( p \) does not depend on \( q \)
- Vector and matrix dimensions are such that all operations are possible.
Calculus of Variations in ME451
[important slide to understand]

- In our case we are interested in changes in kinematic quantities (locations of a point P, of the orientation matrix $A$, etc.) due to variations in the location and orientation of a body

- Variation in location of the LRF:
  \[ r \rightarrow r + \delta r \]

- Variation in orientation of the LRF:
  \[ \phi \rightarrow \phi + \delta \phi \]

- As far as the change of orientation matrix $A(\phi)$ is concerned, using the result stated a couple of slides ago, we have that a variation in the orientation leads to the following variation in $A$:
  \[ \delta A = \frac{\partial A}{\partial \phi} \cdot \delta \phi = B \cdot \delta \phi \]
Change in $A$ due to small variation in $\phi$

[another take]

- Another way to look at it: when I had $\phi$, the orientation matrix looked like this:

$$A = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix}$$

- If the orientation angle went from $\phi$ to $\phi + \delta \phi$, then the orientation matrix went from $A$ to $A + \delta A$, where $\delta A$ is computed as

$$\delta A = B \delta \phi$$
Virtual Displacement of a Point Attached to a Rigid Body

The original position of point $P$:

$$q = \begin{bmatrix} r \\ \phi \end{bmatrix} \Rightarrow r^P(q) = r + As'^P$$

A variation in the generalized coordinates (a variation in the location and orientation of the body) induces a virtual displacement of point $P$:

$$q \mapsto q + \delta q \Rightarrow r^P \mapsto r^P + \delta r^P$$

$$r^P + \delta r^P = (r + \delta r) + (A + \delta A)s'^P \Rightarrow \delta r^P = \delta r + \delta As'^P = \delta r + (B\delta \phi)s'^P$$

Alternatively, use:

$$r^P_q = \begin{bmatrix} I & Bs'^P \end{bmatrix} \Rightarrow \delta r^P = r^P_q \delta q = \begin{bmatrix} I & Bs'^P \end{bmatrix} \begin{bmatrix} \delta r \\ \delta \phi \end{bmatrix} = \delta r + \delta \phi Bs'^P$$
6.1.1

Variational EOM for a Single Rigid Body
Newton’s Laws of Motion

- **1st Law**
  Every body perseveres in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by forces impressed.

- **2nd Law**
  A change in motion is proportional to the motive force impressed and takes place along the straight line in which that force is impressed.

- **3rd Law**
  To any action there is always an opposite and equal reaction; in other words, the actions of two bodies upon each other are always equal and always opposite in direction.

- Newton’s laws
  - are applied to particles (idealized single point masses)
  - only hold in inertial frames
  - are valid only for non-relativistic speeds

Isaac Newton (1642 – 1727)
Body as a Collection of Particles

- Our toolbox provides a relationship between forces and accelerations (Newton’s 2nd law) – but that applies for particles only

- How do we go from particles to bodies?
  - Look at a body as a collection of infinitesimal particles

- Consider a differential mass $dm(P)$ at each point $P$ on the body (located by $s^P$)

- For each such particle, we can write

  $$dm(P)\ddot{x}^P = \sum \text{all forces acting on } dm(P)$$

- What forces should we include?
  - Distributed forces
  - Internal interaction forces, between any two points on the body
  - Concentrated (point) forces
Three Types of Forces Acting on a Differential Mass \( dm(P) \)

1) External distributed forces
- Described using a "force per unit mass":
  \[
  \mathbf{f}_d(P) dm(P)
  \]
- This type of force is not common in classical multibody dynamics
  - One exception: gravitational forces for which \( \mathbf{f}_d(P) = g \)

2) Applied (external) forces
- Concentrated at point \( P \)
- For now, we ignore them and revisit later

3) Internal interaction forces
- Act between point \( P \) and any other point \( R \) on the body, described using a force per units of mass at points \( P \) and \( R \)
- Including the contribution at point \( P \) of all points \( R \) on the body
  \[
  \int_m \mathbf{f}_i(P, R) dm(P) dm(R) = \left( \int_m \mathbf{f}_i(P, R) dm(R) \right) dm(P)
  \]
Newton’s EOM for a Differential Mass \( dm(P) \)

- Apply Newton’s 2\textsuperscript{nd} law to the differential mass \( dm(P) \) located at point \( P \):
  \[
  \ddot{r}^P dm(P) = f_d(P) dm(P) + \left( \int_m f_i(P, R) dm(R) \right) dm(P)
  \]

- This represents a valid way of characterizing the motion of a body: describe the motion of every single particle that makes up that body.

- However
  - Differential equation above involves the internal forces acting within the body.
    - These are difficult to completely describe.
    - Their number is enormous.

- Idea: simplify these equations taking advantage of the \textbf{rigid body} assumption.
The Rigid Body Model

- In the rigid body model, the distance between any pair of points in/on the body is constant:
  
  \[(r^P - r^R)^T (r^P - r^R) = \text{const}\]

- The internal forces

  \[f_i(P, R)dm(R)dm(P) \text{ on } dm(P) \text{ due to the differential mass } dm(R)\]

  \[f_i(R, P)dm(P)dm(R) \text{ on } dm(R) \text{ due to the differential mass } dm(P)\]

  satisfy the following conditions:

  - They act along the line connecting points \(P\) and \(R\)
  
  - They are equal in magnitude, opposite in direction, and collinear

  \[f_i(P, R)dm(R)dm(P) = -f_i(R, P)dm(P)dm(R)\]
The Rigid Body Assumption: Consequences

- The distance between any two points $P$ and $R$ on a rigid body is constant in time:
  \[
  (\mathbf{r}^P - \mathbf{r}^R)^T (\mathbf{r}^P - \mathbf{r}^R) = \text{const}
  \]
  and therefore
  \[
  2 (\mathbf{r}^P - \mathbf{r}^R)^T \delta(\mathbf{r}^P - \mathbf{r}^R) = 0
  \]

- The internal force $\mathbf{f}_i(P, R)dm(P)dm(R)$ acts along the line between $P$ and $R$ and therefore is also orthogonal to $\delta(\mathbf{r}^P - \mathbf{r}^R)$:
  \[
  \mathbf{f}_i^T(P, R)\delta(\mathbf{r}^P - \mathbf{r}^R) = 0 \iff \delta(\mathbf{r}^P - \mathbf{r}^R)^T \mathbf{f}_i(P, R) = 0
  \]
Let $s \in \mathbb{R}^n$. Then $s = 0$ if and only if $v^T s = 0$ for all $v \in \mathbb{R}^n$.

Proof.

$s = 0 \Rightarrow v^T s = 0, \forall v$

This is a trivial consequence of the definition of the dot product (in other words, any vector is perpendicular to the zero vector).

$v^T s = 0, \forall v \Rightarrow s = 0$

Since $v^T s = 0$ for any vector $v$, it is true for the vector $e_i = [0, \cdots, 1, \cdots, 0]$ which has a 1 in the $i$-th position.

Therefore the $i$-th component of $s$ must be zero: $s^T e_i = s_i = 0$.

Repeating this argument for $i = 1, 2, \ldots, n$, we obtain that $s = 0$. 

Variational EOM for a Rigid Body (1)

Start with Newton’s EOM for the differential mass \( dm(P) \):

\[
\ddot{r}^P dm(P) - f_d(P) dm(P) - \left( \int_m f_i(P, R) dm(R) \right) dm(P) = 0
\]

Using the orthogonality theorem, we must have

\[
(\delta r^P)^T \left( \ddot{r}^P dm(P) - f_d(P) dm(P) - \int_m f_i(P, R) dm(R) dm(P) \right) = 0
\]

for arbitrary \( \delta r^P \).

Integrate over the total mass of the body (that is, over all points \( P \)):

\[
\int_m (\delta r^P)^T \ddot{r}^P dm(P) = \int_m (\delta r^P)^T f_d(P) dm(P) + \int_m \int_m (\delta r^P)^T f_i(P, R) dm(R) dm(P)
\]

Note: The rigid body assumptions were not used yet! The above equation holds for arbitrary \( \delta r^P \).
Variational EOM for a Rigid Body (2)

Start from

\[ \int_m (\delta r^P)^T \dot{r}^P \, dm(P) = \int_m (\delta r^P)^T f_d(P) \, dm(P) + \int_m \int_m (\delta r^P)^T f_i(P, R) \, dm(R) \, dm(P) \]

which holds for arbitrary \( \delta r^P \).

Use the **rigid body assumptions** and manipulate the double integral.

End with

\[ \int_m (\delta r^P)^T \ddot{r}^P \, dm(P) = \int_m (\delta r^P)^T f_d(P) \, dm(P) \]

which must hold for all \( \delta r^P \) that are **consistent** with the rigid-body assumptions.
[Side Trip]
D’Alembert’s Principle

For consistent virtual displacements $\delta r^P$

$$\int m (\delta r^P)^T \ddot{r}^P dm(P) = \int m (\delta r^P)^T f_d(P) dm(P)$$

This is D’Alembert’s principle for the motion of a rigid body. D’Alembert’s principle is an extension of the Principle of Virtual Work to the case of accelerated motion.

The two principles allow the elimination of constraint forces when dealing with static or dynamic equilibrium, at the cost of being restricted to consistent virtual displacements.

Jean-Baptiste d’Alembert (1717–1783)
[Side Trip]

**PVW: Simple Statics Example**

Find the force $Q$ such that the structure is in equilibrium at a given angle $\theta$.

"Classical" approach:

- 9 unknowns: $A_x, A_y, B_x, B_y, C_x, C_y, D_x, D_y, Q$
- 9 equations: 3 bodies $\times \{2$ force eqs. $+ 1$ moment eq.$\}$

**PVW**:

$$P \cdot \left( \frac{1}{2} \delta s_B \cdot \cos \theta \right) + F \cdot (\delta s_B \cdot \cos \theta) - Q \cdot \left( \frac{1}{2} \delta s_B \cdot \sin \theta \right) = 0$$

$$\Rightarrow \quad Q = (P + 2F) \cdot \cot \theta$$