ME451
Kinematics and Dynamics of Machine Systems

Elements of 2D Kinematics

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- Quote of the day: “It's hard to beat a person who never gives up.” — Babe Ruth

Before we get started…

- Last time
  - Wrapped up absolute constraints
  - Discussed kinematic relative constraints and how they are expressed in equations
- Today
  - More on relative constraints: translational, composite relative constraints, cam-follower
- Miscellaneous
  - Next week:
    - ADAMS session covered by Dr. Justin Madsen on Tu
    - Usual lecture covered by Dr. Arman Pazouki on Th
    - Dan out of town, for one week – no office hours but I'm monitoring the forum
- HW: Due on Th, 10/09, at 9:30 am
    - Haug's book: 3.3.4, 3.3.5
    - MATLAB 3: available online at class website
  - Post questions on the forum
  - Drop MATLAB assignment in learn@uw dropbox

Revolute Joint

- Step 1: Physically imposes the condition that point P on body *i* and a point P on body *j* are coincident at all times
- Step 2: Identify $\mathbf{\Phi}^{r(i,j)} = 0$
- Step 3: $\mathbf{\Phi}_{\mathbf{q}}^{r(i,j)} = ?$
- Step 4: $\mathbf{\nu}^{r(i,j)} = ?$
- Step 5: $\mathbf{\gamma}^{r(i,j)} = ?$

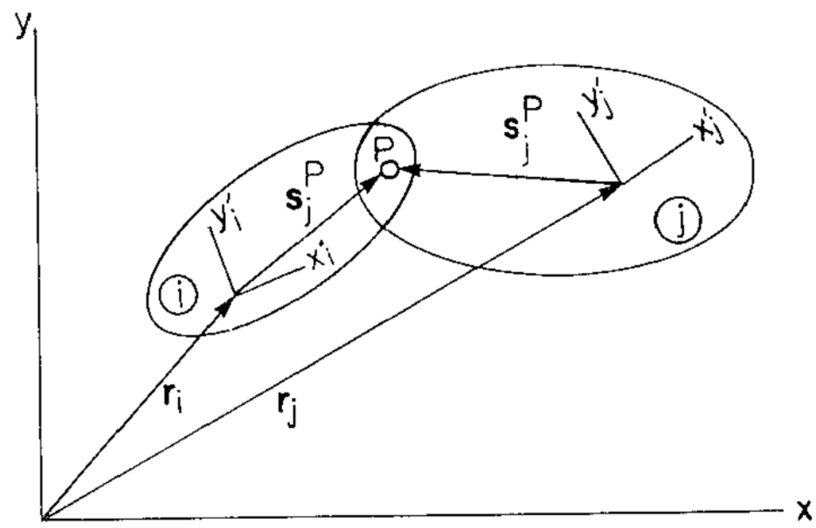


Figure 3.3.4 Revolute joint.

Relative distance-constraint

- Step 1: The distance between the points P_j (on body j) and P_i (on body i) should stay constant and equal to some known value C_4

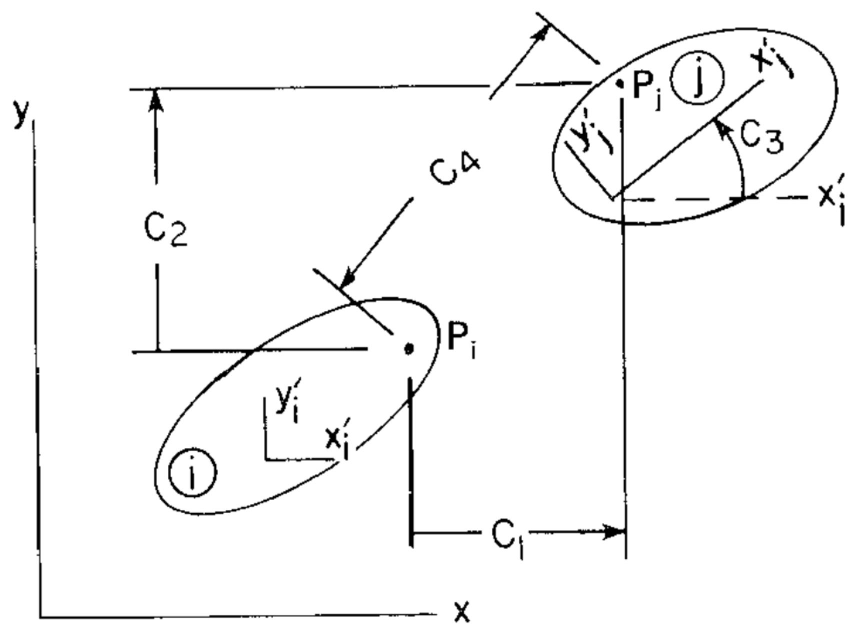


Figure 3.3.1 Simple constraints.

- Step 2: Identify $\Phi^{rd(i,j)} = 0$
- Step 3: $\Phi_{\mathbf{q}}^{rd(i,j)} = ?$
- Step 4: $\nu^{rd(i,j)} = ?$
- Step 5: $\gamma^{rd(i,j)} = ?$

[handout]

Example 3.3.2 / Problem 3.3.3

Four different ways of modeling the same mechanism for Kinematic Analysis

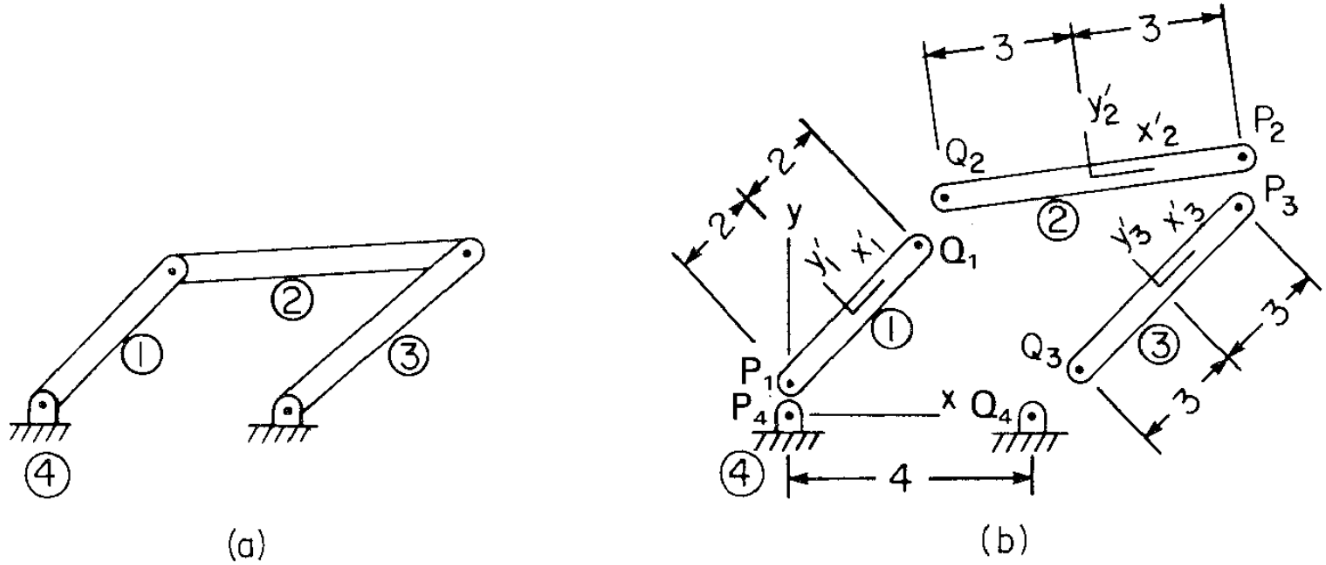


Figure 3.3.2 Four-bar mechanism. (a) Assembled. (b) Dissected.

- Approach 1: bodies 1, 2, and 3
- Approach 2: bodies 1 and 3
- Approach 3: bodies 1 and 2
- Approach 4: body 2

Errata

- Page 67 (sign)

For the translational
be specified on a line tha
between bodies i and j. No
P_j and Q_j are located on bc
vector $\mathbf{v}_j$ in body j con
$\mathbf{v}_i' = [x_i^P - x_i^Q, y_i^P - y_i^Q]^T$ an
on body j. The vector $\mathbf{d}_{ij}$
Vectors $\mathbf{v}_i$ and $\mathbf{v}_j$ must rem
collinear, it is necessary
perpendicular to $\mathbf{v}_i$. Using

$$\mathbf{\Phi}^{t(i,j)} = \begin{bmatrix} (\mathbf{v}_i^{\perp})^T \mathbf{d}_{ij} \\ (\mathbf{v}_i^{\perp})^T \mathbf{v}_j \end{bmatrix}$$

- Page 68 (unbalanced parentheses)

Using Eqs. 2.4.12 and 2.6.8,

$$\boldsymbol{\gamma}^{t(i,j)} = -\begin{bmatrix} \mathbf{v}_i'^T[\mathbf{B}_{ij}\mathbf{s}_j'^P(\dot{\phi}_j - \dot{\phi}_i)^2 - \mathbf{B}_i^T(\mathbf{r}_j - \mathbf{r}_i)\dot{\phi}_i^2 - 2\mathbf{A}_i^T(\dot{\mathbf{r}}_j - \dot{\mathbf{r}}_i)\dot{\phi}_i \\ 0 \end{bmatrix}$$

where the second term on the right is zero, because of Eq. 3.3.13.

Translational Joint

- Step 1: Physically, it allows relative translation between two bodies along a *common axis*. No relative rotation is allowed.
- Step 2: Identify $\mathbf{\Phi}^{t(i,j)} = 0$
- Step 3: $\mathbf{\Phi}_{\mathbf{q}}^{t(i,j)} = ?$
- Step 4: $\boldsymbol{\nu}^{t(i,j)} = ?$
- Step 5: $\boldsymbol{\gamma}^{t(i,j)} = ?$

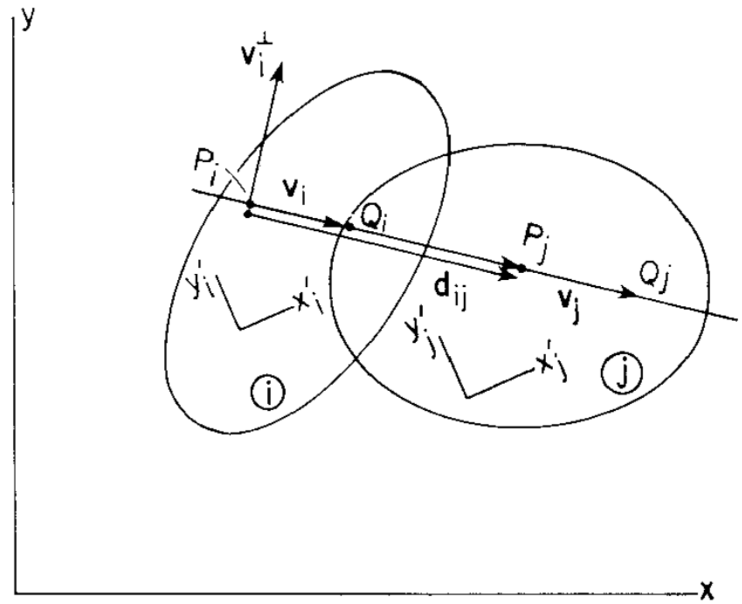


Figure 3.3.5 Translational joint.

Inputs. Attributes. Outputs.

- When you do Kinematics Analysis, i.e., figure out to motion of a mechanism, what are the inputs, the attributes, and the outputs?

- Inputs:
  - The position and orientation of every single body in the mechanism at time t
    - That is, of the reference frame rigidly attached to the body

- Attributes:
  - The mass of each body, the position of the revolute joints on each body, of the translational joints, etc.

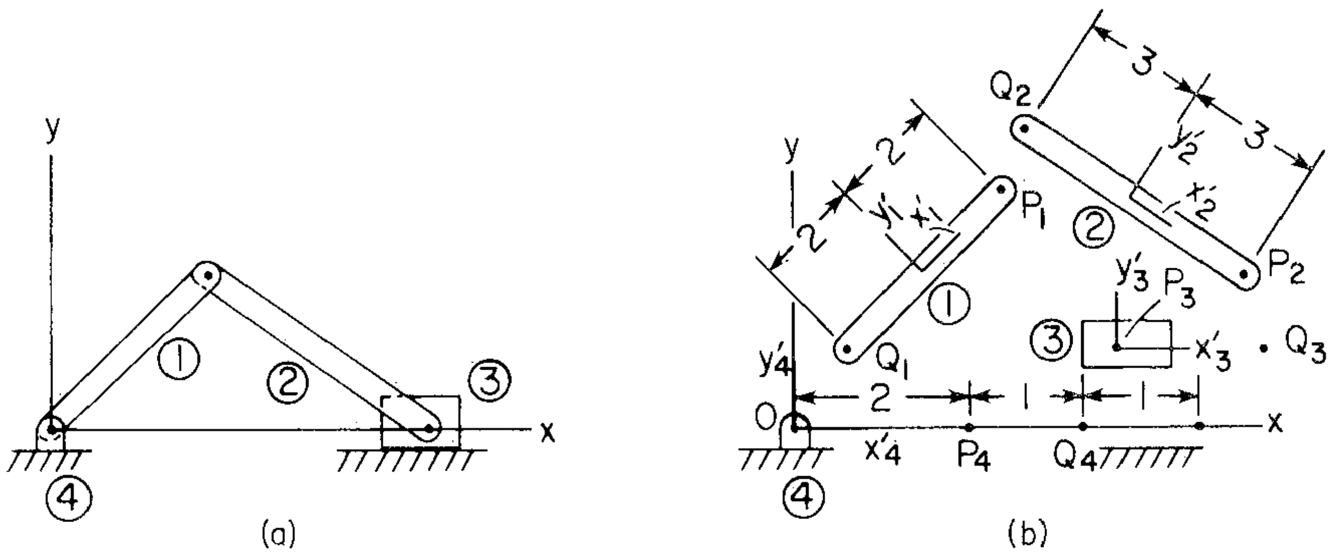
- Outputs:
  - Intermediate: the algebraic equation(s), Jacobian, ν, and , γ
  - Actual: the position and orientation of every single body in the mechanism $t + \delta t$
    - The value δt is some small value, say 0.01 seconds

Kinematic Constraint Attributes

- Examples of constraint attributes:
  - For a revolute joint:
    - You know where the joint is located, so therefore you know $\mathbf{s}_i'^P$
  - For a translational join:
    - You know what the direction of relative translation is, so therefore you know $\mathbf{v}_j'$
  - For a distance constraint:
    - You know the distance C_4

Example 3.3.4

- Consider the slider-crank below. Come up with the set of kinematic constraint equations to kinematically model this mechanism

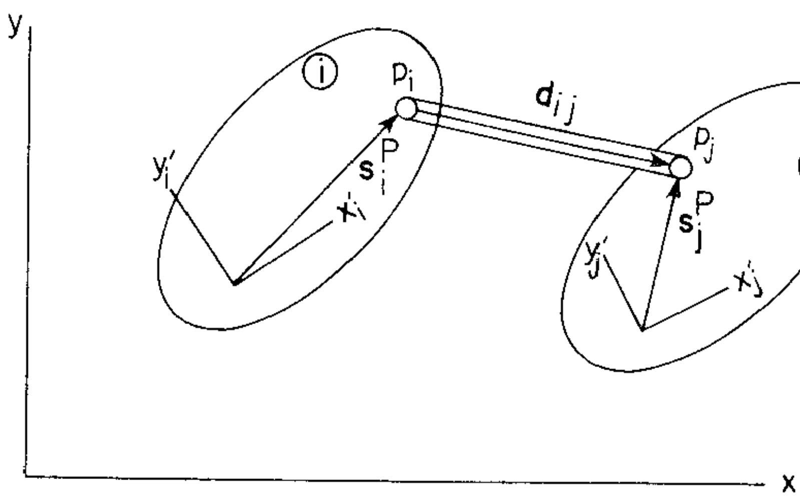


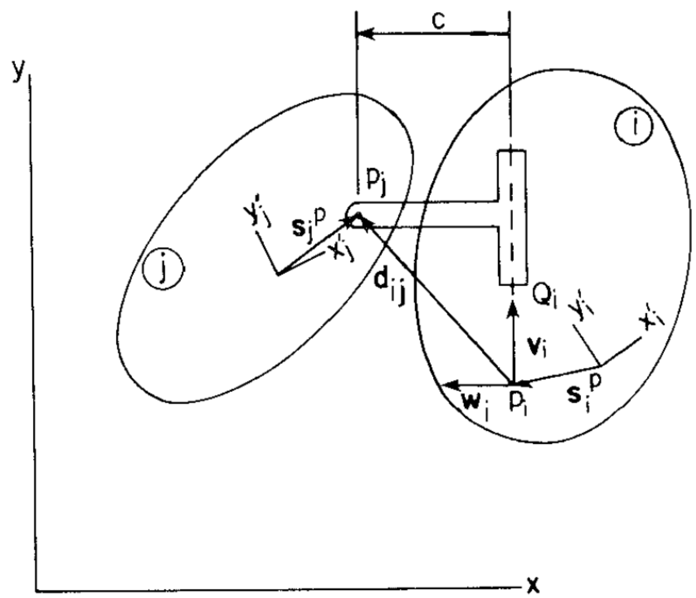
Composite Joints (1)

- Just a means to eliminate one intermediate body (a.k.a. coupler) whose kinematics you are not interested in

- Revolute-Revolute
  - Eliminates need of connecting rod
- Attributes:
  - Points P_i and P_j: $\mathbf{s}'^P_i$ and $\mathbf{s}'^P_j$
  - Length of the massless rod: $\|\vec{\mathbf{d}}_{ij}\|$

- Revolute-Translational
  - Eliminates the intermediate body
- Attributes:
  - Distance c
  - Point P_j (location of revolute joint)
  - Axis of translation: $\mathbf{v}'_i$





Composite Joints (2)

- One follows exactly the same steps as for any other joint:
  - Step 1: Physically, what type of motion does the joint allow?
  - Step 2: Identify $\mathbf{\Phi}^{(i,j)} = 0$
  - Step 3: $\mathbf{\Phi}_{\mathbf{q}}^{(i,j)} = ?$
  - Step 4: $\boldsymbol{\nu}^{(i,j)} = ?$
  - Step 5: $\boldsymbol{\gamma}^{(i,j)} = ?$

3.4.1

GEARS

(CONVEX-CONVEX, CONCAVE-CONVEX, RACK AND PINION)

Gears

- Convex-convex gears
  - Gear teeth on the periphery of the gears cause the pitch circles shown to roll relative to each other, without slip
- First Goal: find the angle θ , that is, the angle of the carrier

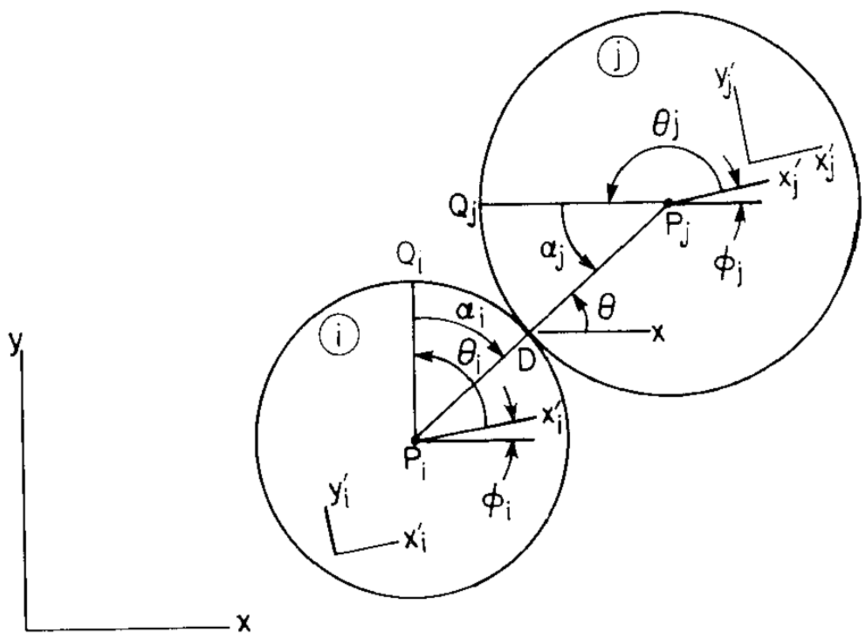


Figure 3.4.2 Geometry of gear set.

- What's known:
  - Angles θ_i and θ_j
  - The radii R_i and R_j
- You need to express θ as a function of these four quantities plus the orientation angles ϕ_i and ϕ_j
- Kinematically: P_iP_j should always be perpendicular to the contact plane

Gears - Discussion of Figure 3.4.2 (Geometry of gear set)

$O'_i x'_i y'_i$ represents the local reference frame attached to gear i. This reference frame is rotated by an angle ϕ_i with respect to the global reference frame. This angle ϕ_i depends on time, and changes as the attitude (orientation) of body i changes.

$O'_j x'_j y'_j$ represents the local reference frame attached to gear j. This reference frame is rotated by an angle ϕ_j with respect to the global reference frame. This angle ϕ_j depends on time, and changes as the attitude (orientation) of body j changes.

When the gears were assembled, the points Q_i and Q_j were the contact points between the two gears (that's where the gears came in contact for the first time, at time $t = 0$). If one gear is activated and it starts rotating, the second gear will follow and the two points Q_i and Q_j will separate and follow their destiny, away from the contact point which is denoted by D.

There is an angle α_i that indicates how far the original contact point, Q_i, is from the current contact point, D. By the same token, there is an angle α_j that indicates how far the original contact point, Q_j, is from the current contact point, D.

Since the local reference frame $O'_i x'_i y'_i$ is attached to gear i, and the point Q_i is also attached to gear i, the angle that positions Q_i in $O'_i x'_i y'_i$ stays constant at all times, and it's denoted by θ_i. For gear j a similar angle θ_j is defined.

Gears - Discussion of Figure 3.4.2 (Geometry of gear set)

The contact point D is always on the line that connects P_i and P_j, the centers of the two gears, respectively.

What we are after is determining the angle θ that implicitly defines the perpendicular on the plane of contact (the plane that goes through D and is tangent to the two gears at point D). Once this angle θ becomes available, the vector that is perpendicular on the tangent plane assumes the expression:

$$\mathbf{u}^{\perp} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}.$$

Using Figure 3.4.2, after performing some manipulations, the angle θ is expressed as

$$\theta = \frac{R_i(\phi_i + \theta_i) + R_j(\phi_j + \theta_j - \pi)}{R_i + R_j}. \tag{1}$$

The kinematic constraint associated with the gear set requires that the vectors $\vec{P_i P_j}$ and $\vec{\mathbf{u}}$ are parallel, or in other words,

$$\Phi^{g(i,j)} = (\mathbf{r}_j^P - \mathbf{r}_i^P)^T \cdot \mathbf{u}^{\perp} = 0 \tag{2}$$

Note that the important thing is that this angle θ depends on the value of ϕ_i and ϕ_j, which in turn depend on the orientation of the two gears. What Eq. (2) is telling us is that ϕ_i and ϕ_j can not be arbitrarily changing. Rather, as they change in time, they should change in such a way so that the angle θ computed with Eq. (1) will satisfy the condition of Eq. (2).

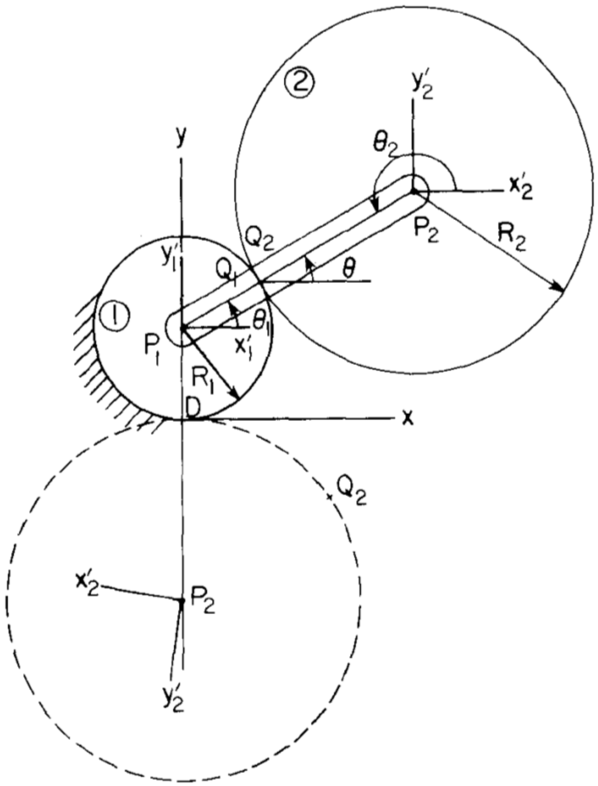
Note: there are a couple of mistakes in the book, see Errata slide

Gear Set Constraints

- Step 1: Understand the physical joint
- Step 2: $\mathbf{\Phi}^{g(i,j)} = \left(\mathbf{r}_j^P - \mathbf{r}_i^P\right)^T \mathbf{u}^{\perp} = \mathbf{0}$
- Step 3: $\mathbf{\Phi}_{\mathbf{q}}^{g(i,j)} = ?$
- Step 4: $\nu^{g(i,j)} = ?$
- Step 5: $\gamma^{g(i,j)} = ?$

Example: 3.4.1

- Gear 1 is fixed to ground
- Given to you: $\phi_1 = 0$, $\theta_1 = \pi/6$, $\theta_2 = 7\pi/6$, $R_1 = 1$, $R_2 = 2$
- Find ϕ_2 as gear 2 falls to the position shown (carrier line P_1P_2 becomes vertical)



Gears (Convex-Concave)

- Convex-concave gears – we are not going to look into this class of gears
- The approach is the same, that is, expressing the angle θ that allows on to find the angle of the
- Next, a perpendicularity condition using **u** and P_iP_j is imposed (just like for convex-convex gears)

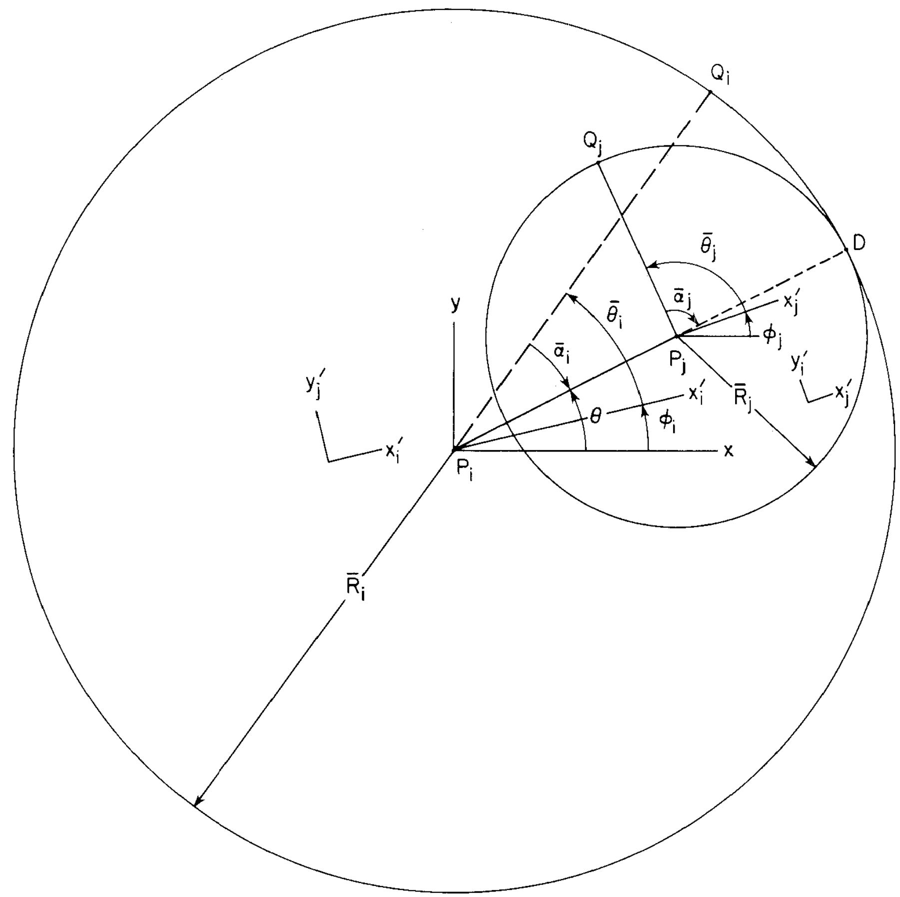


Figure 3.4.4 Concave–convex gear set.

Rack and Pinion Preamble

- Framework:
  - Two points P_i and Q_i on body i define the rack center line
  - Radius of pitch circle for pinion is R_j
  - There is no relative sliding between pitch circle and rack center line
  - Q_i and Q_j are the points where the rack and pinion were in contact at time t=0

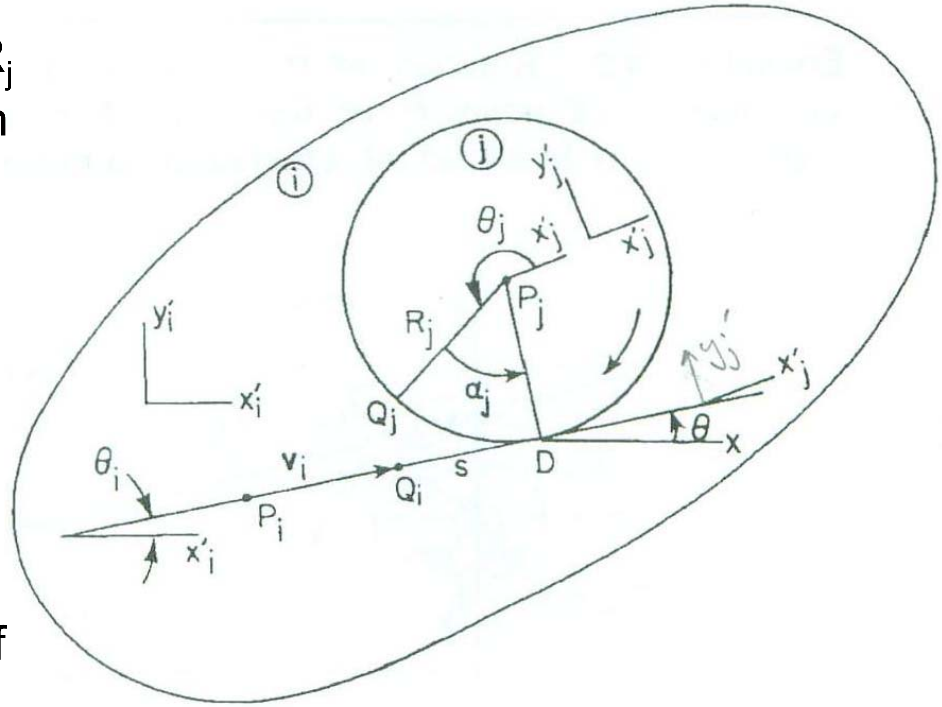


Figure 3.4.5 Rack and pinion.

- NOTE:
  - A rack-and-pinion type kinematic constraint is a limit case of a pair of convex-convex gears
    - Take the radius R_i to infinity, and the pitch line for gear i will become the rack center line

Rack and Pinion Kinematics

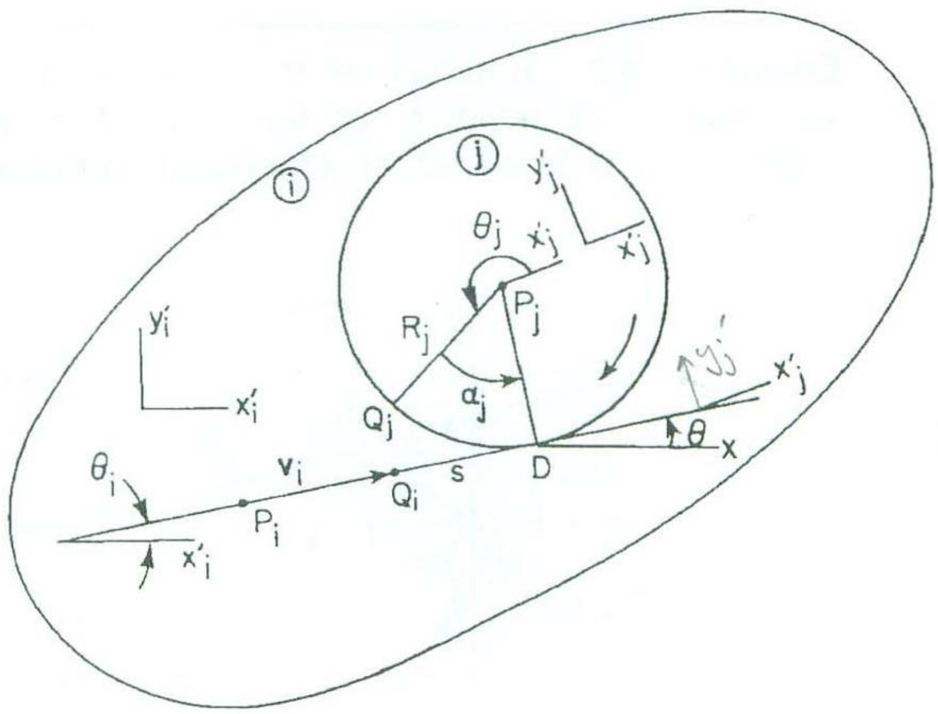


Figure 3.4.5 Rack and pinion.

- Kinematic constraints that define the relative motion:
  - At any time, the distance between the point P_j and the contact point D should stay constant (this is equal to the radius of the gear R_j)
  - The length of the segment Q_iD and the length of the arc Q_jD should be equal (no slip condition)

- Rack-and-pinion removes two DOFs of the relative motion between these two bodies

Rack and Pinion Constraints

- Step 1: Understand the physical joint
- Step 2: $\mathbf{\Phi}^{rp(i,j)} = \begin{bmatrix} \left(\mathbf{r}_j^P - \mathbf{r}_i^Q\right)^T \mathbf{v}_i - v_i R_j \alpha_j \\ \left(\mathbf{r}_j^P - \mathbf{r}_i^P\right)^T \mathbf{v}_i^{\perp} - v_i R_j \end{bmatrix} = \mathbf{0}$
- Step 3: $\mathbf{\Phi}_{\mathbf{q}}^{rp(i,j)} =?$
- Step 4: $\nu^{rp(i,j)} =?$
- Step 5: $\gamma^{rp(i,j)} =?$

Errata:

- Page 73
  (transpose and signs)

$$\Phi^{g(i,j)} = (\mathbf{r}_j^P - \mathbf{r}_i^P)\mathbf{u}^\perp$$
$$= (x_j^P - x_i^P)\sin\theta - (y_j^P - y_i^P)\cos\theta = 0 \quad \textbf{(3.4.3)}$$

where θ is given by Eq. 3.4.2 and $\mathbf{u}^\perp \equiv [-\sin\theta, \cos\theta]^T$; that is, $\mathbf{u} = [\cos\theta, \sin\theta]^T$ is a unit vector along the line from P_i to P_j in Fig. 3.4.2.

- Page 73
  (perpendicular sign,
  both equations)

$$\Phi_{\mathbf{q}_i}^{g(i,j)} = \left[-\mathbf{u}^T, -\mathbf{s}_i'^{P^T}\mathbf{B}_i^T\mathbf{u} + (\mathbf{r}_j^P - \mathbf{r}_i^P)^T\mathbf{u}^\perp\left(\frac{R_i}{R_i + R_j}\right)\right]$$

$$\Phi_{\mathbf{q}_j}^{g(i,j)} = \left[\mathbf{u}^T, \mathbf{s}_j'^{P^T}\mathbf{B}_j^T\mathbf{u} + (\mathbf{r}_j^P - \mathbf{r}_i^P)^T\mathbf{u}^\perp\left(\frac{R_j}{R_i + R_j}\right)\right] \quad \textbf{(3.4.4)}$$