ME451
Kinematics and Dynamics of Machine Systems

Elements of 2D Kinematics

September 30, 2014

- Quote of the day: "The first 90% of the code accounts for the first 90% of the development time. The remaining 10% of the code accounts for the other 90% of the development time." - Tom Cargil

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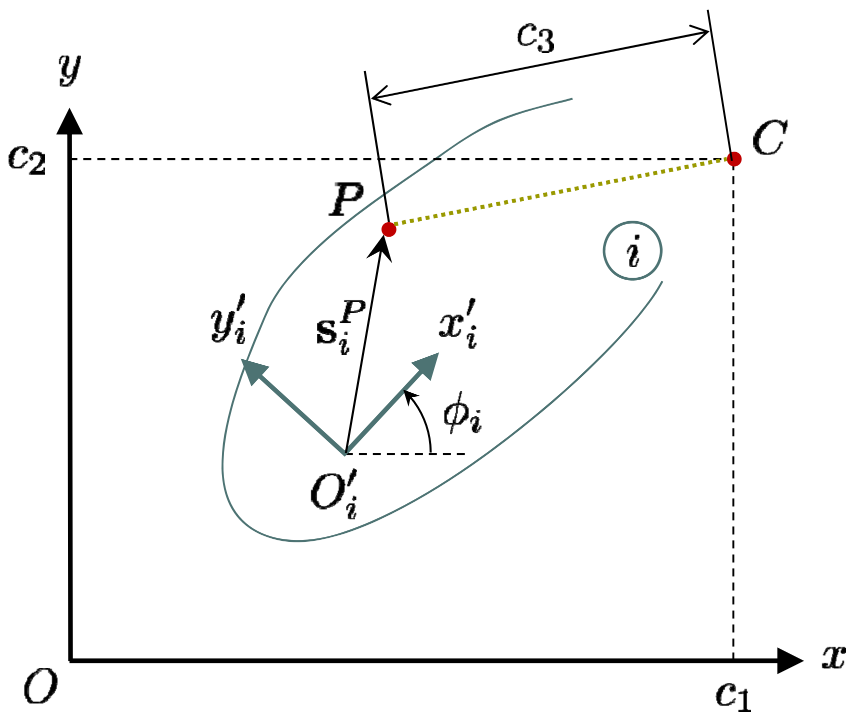
Before we get started…

- Last time
  - Discussed kinematic absolute constraints and how they are expressed in equations
    - Learn how to compute the essential ingredients required by a Kinematics analysis
- Today
  - Wrap up absolute constraints
  - Discuss kinematic relative constraints and how they are expressed in equations
- Miscellaneous
  - Provide anonymous feedback on Th, please use the form emailed out
- HW: Due on Th, 10/02, at 9:30 am
    - Haug's book: 3.1.3, 3.3.2
    - MATLAB 2: available online at class website
  - Post questions on the forum
  - Drop MATLAB assignment in learn@uw dropbox

Absolute distance-constraint

- Step 1: the distance from a point P on body i to a point C defined in the GRF stays constant and equal to some known value c_3



- Step 2: Identify $\Phi^{ad(i)} = 0$
- Step 3: $\Phi_{\mathbf{q}}^{ad(i)} = ?$
- Step 4: $\nu^{ad(i)} = ?$
- Step 5: $\gamma^{ad(i)} = ?$

Example 3.2.1

- An example using absolute coordinate constraints:
  simple pendulum

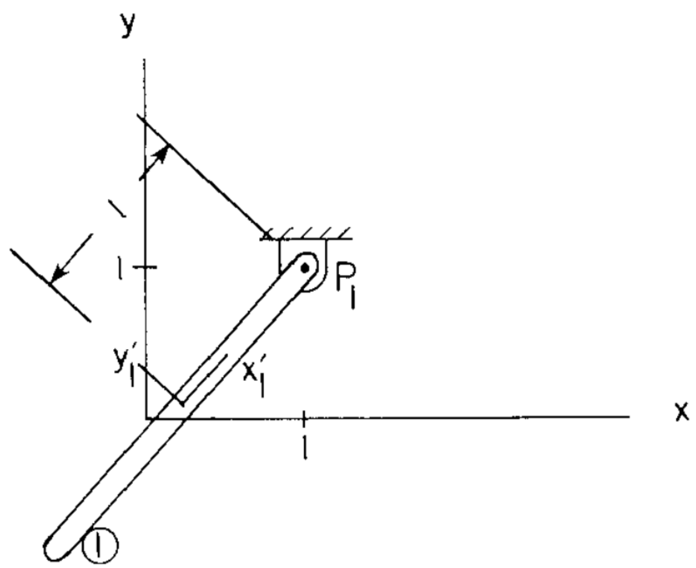


Figure 3.2.3 Simple pendulum with absolute constraints.

$$\mathbf{q} = \begin{bmatrix} x_1 & y_1 & \phi_1 \end{bmatrix}^T$$

$$\mathbf{r}^P = \mathbf{r}_1 + \mathbf{A}_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 + \cos\phi_1 \\ y_1 + \sin\phi_1 \end{bmatrix}$$

$$\boldsymbol{\Phi} \equiv \begin{bmatrix} \boldsymbol{\Phi}^{ax(1)} \\ \boldsymbol{\Phi}^{ay(1)} \end{bmatrix} = \begin{bmatrix} x_1^P - 1 \\ y_1^P - 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{\Phi}_{\mathbf{q}} = \begin{bmatrix} 1 & 0 & -\sin\phi_1 \\ 0 & 1 & \cos\phi_1 \end{bmatrix}$$

$$\boldsymbol{\nu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{\gamma} = \begin{bmatrix} \dot{\phi}_1^2 \cos\phi_1 \\ \dot{\phi}_1^2 \sin\phi_1 \end{bmatrix}$$

Example 3.2.2

- An example using an absolute angle constraint: slider along x-axis

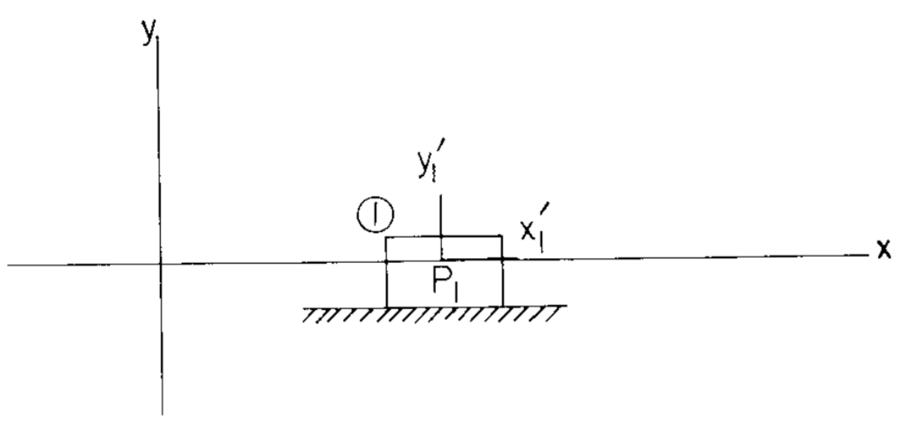


Figure 3.2.4 Slider along x axis.

$$\mathbf{q} = \begin{bmatrix} x_1 & y_1 & \phi_1 \end{bmatrix}^T$$

$$\mathbf{\Phi} \equiv \mathbf{\Phi}^{a\phi(1)} = \phi_1 - 0 = 0$$

$$\mathbf{\Phi_q} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{\nu} = 0$$

$$\boldsymbol{\gamma} = 0$$

Attributes of a Constraint

[it'll be on the exam]

- What do you need to specify to completely define a certain type of constraint?
  - In other words, what are the attributes of a constraint; i.e., the parameters that define it?

- For absolute-x constraint you need to specify: the body "i", the point P on that body, and the value that x_i^P should assume
  - Three pieces of information, that is

- For a distance constraint, you need to specify the "distance", but also the location of point P in the LRF, the body "i" on which the LRF is attached to, as well as the coordinates c_1 and c_2 of point C in the GRF (Global Ref. Frame).

- How about an absolute angle constraint?
  - The body i, and the value of the angle – two pieces of information

[handout]

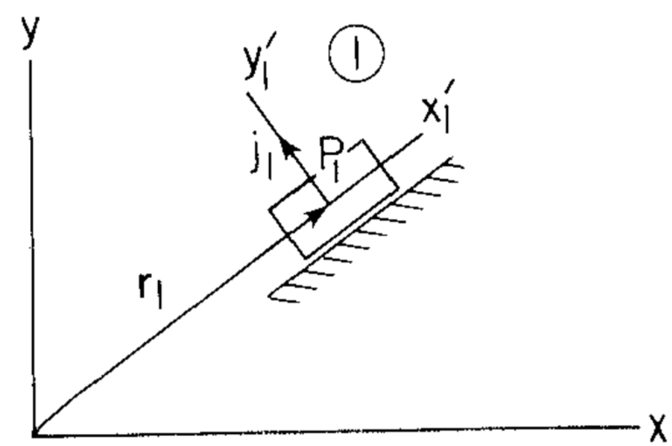
Example 3.1.3

An example where the constraint equations fail to imply the actual kinematics of a mechanism: block sliding on an incline (of angle $\pi/4$).
One way of describing the kinematics (using $\mathbf{q} = [x_1, y_1, \phi_1]^T$):

- Point P_1 remains on the slide line $\Rightarrow x_1 = y_1$
- The y' axis is orthogonal to $\mathbf{r}_1$ (the slide line) $\Rightarrow \mathbf{r}_1^T \mathbf{A}(\phi_1)\mathbf{j}_1' = 0$
- The block moves with prescribed x velocity

In other words, we get:

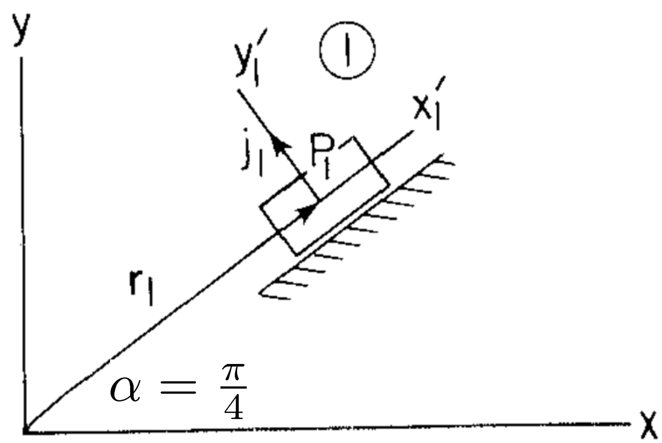
$$\mathbf{\Phi}(\mathbf{q}, t) = \begin{bmatrix} x_1 - y_1 \\ -x_1 \sin\phi_1 + y_1 \cos\phi_1 \\ x_1 - 6 + 6t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

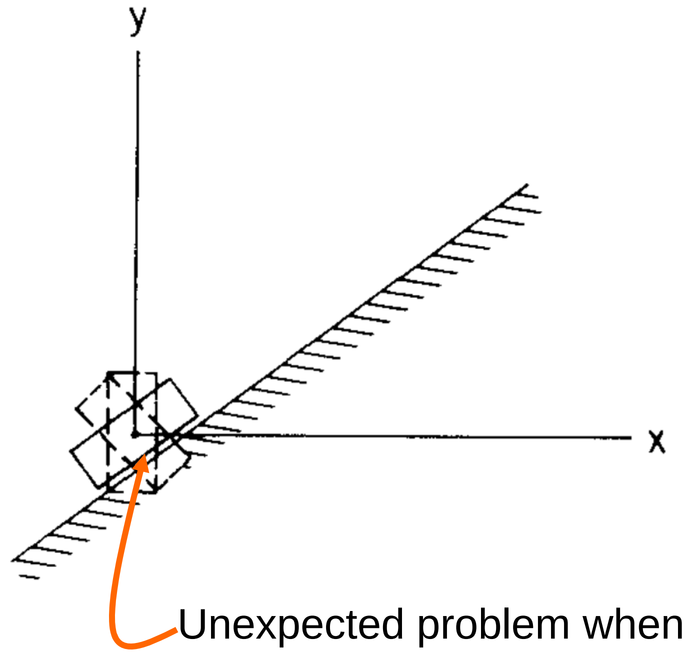


It looks legit, doesn't it?
But what happens when $t = 1$?

Example 3.1.3

- Note that when passing through the origin, the algebraic constraints fail to specify the actual kinematics of the mechanism
- Translating plain English into equations is not always straightforward





Unexpected problem when passing through origin…

- Translating plain English into the right equations:

$$\mathbf{\Phi}(\mathbf{q}, t) = \begin{bmatrix} x_1 - y_1 \\ \phi_1 - \pi/4 \\ x_1 - 6 + 6t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

3.3

RELATIVE CONSTRAINTS

Preamble:

Switching representation between two Reference Frames with rotation matrices A_i and A_j, respectively

Given the representation $\mathbf{s}'_j$ of a vector in a LRF with rotation matrix $\mathbf{A}_j = \mathbf{A}(\phi_j)$, what is the representation $\mathbf{s}'_i$ of the **same** vector in another LRF characterized by $\mathbf{A}_i = \mathbf{A}(\phi_i)$?

The solution can be obtain by going through the GRF:

$$\mathbf{s} = \mathbf{A}_j \mathbf{s}'_j \quad \Rightarrow \quad \mathbf{s}'_i = \mathbf{A}_i^T \mathbf{s} = \mathbf{A}_i^T \mathbf{A}_j \mathbf{s}'_j$$

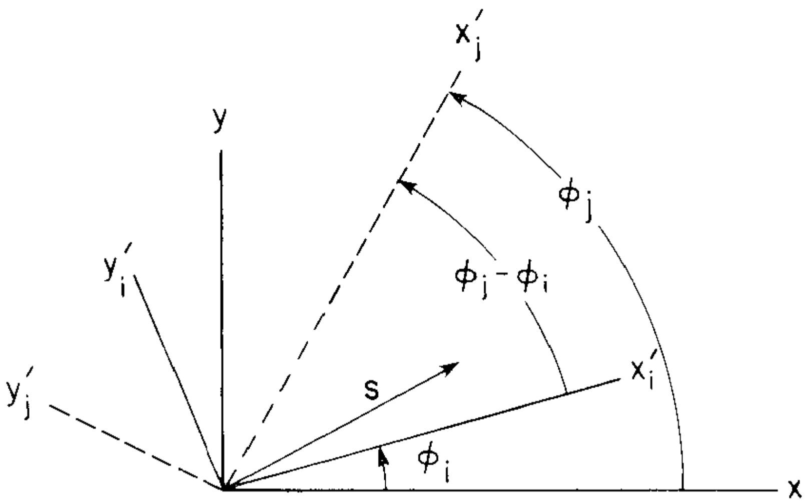


Figure 2.4.4 Three reference frames with coincident origins.

$$\mathbf{s}'_i = \mathbf{A}_i^T \mathbf{A}_j \mathbf{s}'_j$$

Preamble, Continued:

Notation (related to changing representation from A_j to A_i)

- Since

$$\mathbf{A}_i^T \mathbf{A}_j = \begin{bmatrix} \cos\phi_i & \sin\phi_i \\ -\sin\phi_i & \cos\phi_i \end{bmatrix} \cdot \begin{bmatrix} \cos\phi_j & -\sin\phi_j \\ \sin\phi_j & \cos\phi_j \end{bmatrix}$$
$$= \begin{bmatrix} \cos(\phi_j - \phi_i) & -\sin(\phi_j - \phi_i) \\ \sin(\phi_j - \phi_i) & \cos(\phi_j - \phi_i) \end{bmatrix}$$

using the notation $\phi_{ij} \triangleq \phi_j - \phi_i$ we get that

$$\mathbf{A}_i^T \mathbf{A}_j = \mathbf{A}(\phi_j - \phi_i) = \mathbf{A}(\phi_{ij}) \triangleq \mathbf{A}_{ij} \quad \Rightarrow \quad \mathbf{s}_i' = \mathbf{A}_{ij}\mathbf{s}_j'$$

- Note that the order is **important**: it is $\mathbf{A}_{ij}$ and not $\mathbf{A}_{ji}$!
- $\mathbf{A}_{ij}$ is applied to a vector represented in the j reference frame and expresses that same vector in the i frame. In other words, it is $\mathbf{A}_{\{to\}\{from\}}$.
- In this light, when we use $\mathbf{A}_j$, we really mean $\mathbf{A}_{0j}$, where 0 indicates the GRF. Indeed, $\mathbf{A}_j = \mathbf{A}_{0j} = \mathbf{A}(\phi_j - 0) = \mathbf{A}(\phi_j)$.

Preamble, Final Slide

(related to changing representation from A_j to A_i)

- For later reference, it is useful to recall that,

$$\frac{\partial \mathbf{A}(\phi)}{\partial \phi} \equiv \mathbf{B}(\phi) = \mathbf{A}(\phi)\mathbf{R} = \mathbf{RA}(\phi)\,, \text{ where } \mathbf{R} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

- Therefore

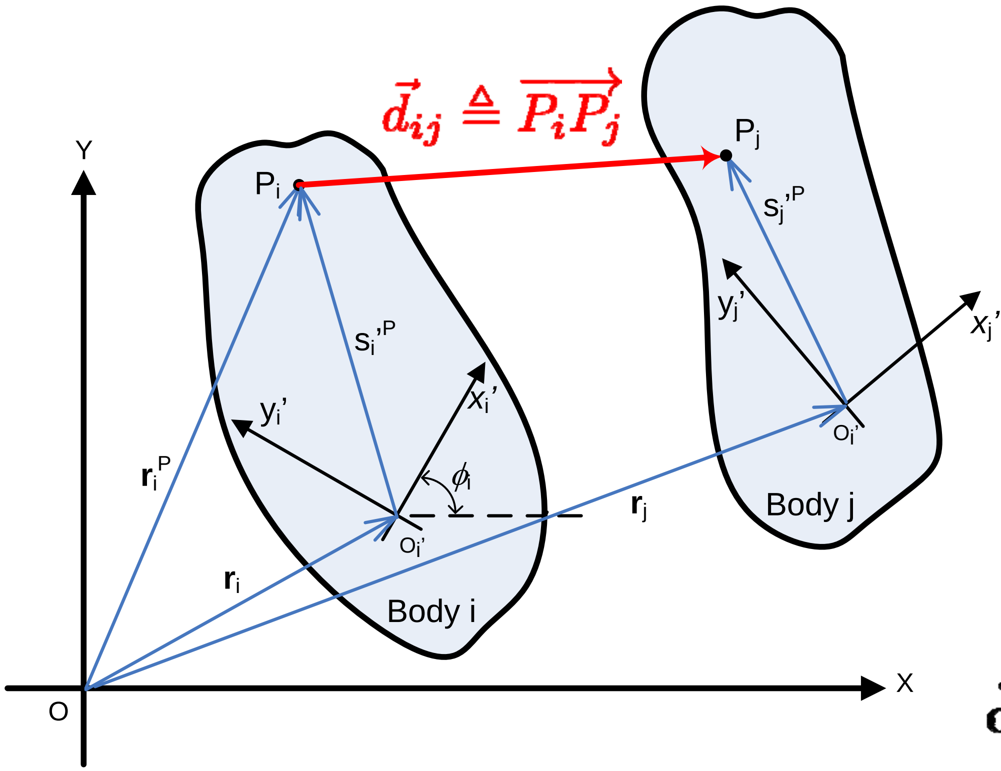
$$\mathbf{B}_{ij} = \mathbf{RA}_{ij} = \mathbf{A}_{ij}\mathbf{R} = \begin{bmatrix} -\sin(\phi_j - \phi_i) & -\cos(\phi_j - \phi_i) \\ \cos(\phi_j - \phi_i) & -\sin(\phi_j - \phi_i) \end{bmatrix}$$

- Based on the definition of A_{ij} (see previous slide) we have that

$$\frac{\partial \mathbf{A}_{ij}}{\partial \phi_j} = \mathbf{B}_{ij} \qquad \frac{\partial \mathbf{A}_{ij}}{\partial \phi_i} = -\mathbf{B}_{ij}$$

Vector between P_i and P_j

- Something that we'll use a lot: the expression of the vector from P_i to P_j in terms of the generalized coordinates **q**



$$\mathbf{q} = \begin{bmatrix} x_i \\ y_i \\ \phi_i \\ x_j \\ y_j \\ \phi_j \end{bmatrix} = \begin{bmatrix} \mathbf{r}_i \\ \phi_i \\ \mathbf{r}_j \\ \phi_j \end{bmatrix} = \begin{bmatrix} \mathbf{q}_i \\ \mathbf{q}_j \end{bmatrix}$$

$$\mathbf{d}_{ij} = \mathbf{r}_j + \mathbf{A}_j \mathbf{s}'^P_j - \mathbf{r}_i - \mathbf{A}_i \mathbf{s}'^P_i$$

$$\dot{\mathbf{d}}_{ij} = \dot{\mathbf{r}}_j + \dot{\phi}_j \mathbf{B}_j \mathbf{s}'^P_j - \dot{\mathbf{r}}_i - \dot{\phi}_i \mathbf{B}_i \mathbf{s}'^P_i$$

$$\ddot{\mathbf{d}}_{ij} = \ddot{\mathbf{r}}_j + \ddot{\phi}_j \mathbf{B}_j \mathbf{s}'^P_j - \dot{\phi}_j^2 \mathbf{A}_j \mathbf{s}'^P_j - \ddot{\mathbf{r}}_i - \ddot{\phi}_i \mathbf{B}_i \mathbf{s}'^P_i + \dot{\phi}_i^2 \mathbf{A}_i \mathbf{s}'^P_i$$

Relative x-constraint

- Step 1: The difference between the x coordinates of point P_j and point P_i should stay constant and equal to some known value C_1

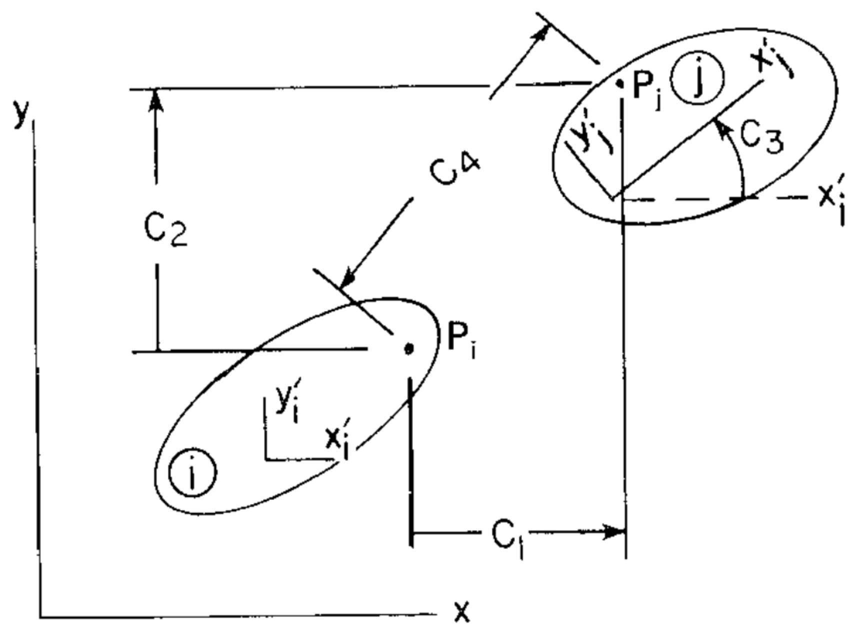


Figure 3.3.1 Simple constraints.

- Step 2: Identify $\Phi^{rx(i,j)} = 0$
- Step 3: $\Phi_{\mathbf{q}}^{rx(i,j)} = ?$
- Step 4: $\nu^{rx(i,j)} = ?$
- Step 5: $\gamma^{rx(i,j)} = ?$

Relative y-constraint

- Step 1: The difference between the y coordinates of point P_j and point P_i should stay constant and equal to some known value C_2

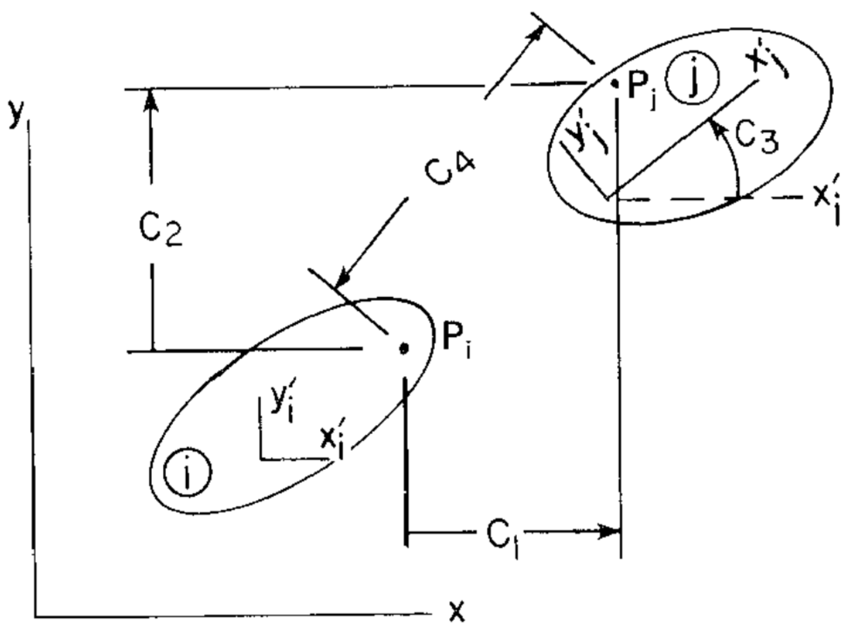


Figure 3.3.1 Simple constraints.

- Step 2: Identify $\Phi^{ry(i,j)} = 0$
- Step 3: $\Phi_{\mathbf{q}}^{ry(i,j)} = ?$
- Step 4: $\nu^{ry(i,j)} = ?$
- Step 5: $\gamma^{ry(i,j)} = ?$

Relative angle-constraint

- Step 1: The difference between the orientation angles of the LRFs associated with bodies i and j should stay constant and equal to some known value C_3

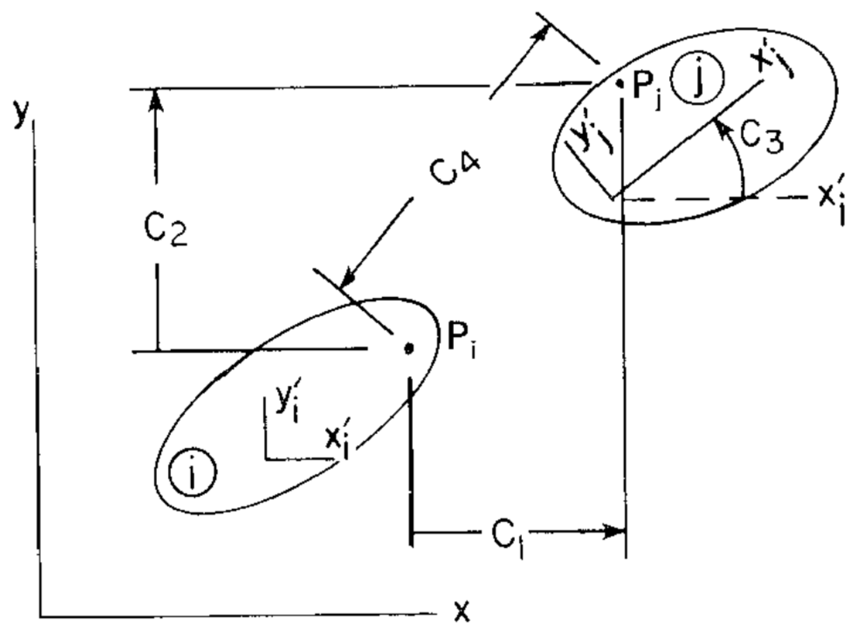


Figure 3.3.1 Simple constraints.

- Step 2: Identify $\Phi^{r\phi(i,j)} = 0$
- Step 3: $\Phi_{\mathbf{q}}^{r\phi(i,j)} = ?$
- Step 4: $\nu^{r\phi(i,j)} = ?$
- Step 5: $\gamma^{r\phi(i,j)} = ?$

Revolute Joint

- Step 1: Physically imposes the condition that point P on body *i* and a point P on body *j* are coincident at all times
- Step 2: Identify $\mathbf{\Phi}^{r(i,j)} = 0$
- Step 3: $\mathbf{\Phi}_{\mathbf{q}}^{r(i,j)} = ?$
- Step 4: $\boldsymbol{\nu}^{r(i,j)} = ?$
- Step 5: $\boldsymbol{\gamma}^{r(i,j)} = ?$

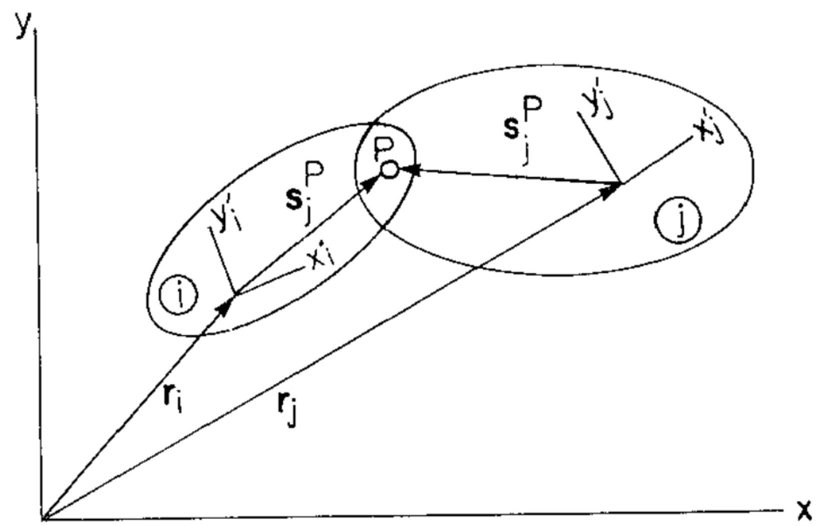


Figure 3.3.4 Revolute joint.