Before we get started...

- **Last time**
  - Time derivatives
  - Partial derivatives (hard, messy, and widely used in ME451)

- **Today**
  - Wrap up partial derivatives (sensitivity computation)
    - Focus is on chain rule

- **HW: assigned last time**
  - Due on Th, 9/18, at 9:30 am
  - Problems assigned in class and 2.4.4, 2.5.2, 2.5.7 out of Haug’s book
  - Post questions on the forum
Complex Case 1
Scalar Function of Vector Variable

- \( f \) is a scalar function of “n” variables: \( q_1, \ldots, q_n \)

\[
f : \mathbb{R}^n \to \mathbb{R}
\]

- However, each of these variables \( q_i \) in turn depends on a set of “k” other variables \( x_1, \ldots, x_k \).

\[
q : \mathbb{R}^k \to \mathbb{R}^n, \quad q \triangleq q(x) = \begin{bmatrix} q_1(x_1, \ldots, x_k) \\ \vdots \\ q_n(x_1, \ldots, x_k) \end{bmatrix}
\]

- The composition of \( f \) and \( q \) leads to a new function:

\[
\phi : \mathbb{R}^k \to \mathbb{R}, \quad \phi(x) = (f \circ q)(x) \triangleq f(q(x))
\]
Chain Rule
Scalar Function of Vector Variable

- Question: how do you compute $\phi_x$?

- Using our notation:
  \[ \phi(x) = (f \circ q)(x) = f(q(x)) \quad \Rightarrow \quad \phi_x =? \]

- Chain Rule:
  \[
  \phi_x \equiv \frac{\partial \phi}{\partial x} = \frac{\partial f}{\partial q} \cdot \frac{\partial q}{\partial x} \equiv f_q \cdot q_x
  \]

\[
\phi_x \equiv \begin{pmatrix} \frac{\partial f}{\partial q} \cdot q_x \end{pmatrix}_{1 \times k}
\]

\[
\begin{pmatrix} \frac{\partial f}{\partial q} \cdot q_x \end{pmatrix}_{1 \times k}
\begin{pmatrix} q_x \end{pmatrix}_{n \times 1}
\begin{pmatrix} f_q \end{pmatrix}_{1 \times n}
\begin{pmatrix} q_x \end{pmatrix}_{n \times 1}
\begin{pmatrix} f_q \cdot q_x \end{pmatrix}_{1 \times k}
\]
Assume that $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and consider a function $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined as $\phi(y) = 3y_1^2 + \sin y_2$. Assume further that $y$ depends on a variable $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ as follows:

$$y \triangleq y(x) \equiv \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} 2x_1 + \log_{10} x_2 + \sqrt{x_3} \\ (x_1 - x_2)^2 \end{bmatrix}$$

It follows that $\phi$ depends on $x$, implicitly through $y$. Apply the chain rule of differentiation to find the derivative of $\phi$ with respect to $x$, that is:

$$\phi_x \triangleq \begin{bmatrix} \frac{\partial \phi}{\partial x_1} \\ \frac{\partial \phi}{\partial x_2} \\ \frac{\partial \phi}{\partial x_3} \end{bmatrix} = ?$$

What is the dimension of the Jacobian (sensitivity matrix) $\phi_x$?
Complex Case 2
Vector Function of Vector Variable

- \( F \) is a vector function of several variables: \( q_1, \ldots, q_n \)
  \[
  F : \mathbb{R}^n \rightarrow \mathbb{R}^m
  \]

- However, each of these variables \( q_i \) depends in turn on a set of \( k \) other variables \( x_1, \ldots, x_k \).
  \[
  q : \mathbb{R}^k \rightarrow \mathbb{R}^n, \quad q \triangleq q(x) = \begin{bmatrix}
  q_1(x_1, \ldots, x_k) \\
  \vdots \\
  q_n(x_1, \ldots, x_k)
  \end{bmatrix}
  \]

- The composition of \( F \) and \( q \) leads to a new function:
  \[
  \Phi : \mathbb{R}^k \rightarrow \mathbb{R}^m, \quad \Phi(x) = (F \circ q)(x) \triangleq F(q(x))
  \]
Question: how do you compute $\Phi_x$?

Using our notation:

$$\Phi(x) = (F \circ q)(x) = F(q(x)) \Rightarrow \Phi_x =?$$

Chain Rule:

$$\Phi_x = \frac{\partial \Phi}{\partial x} = \frac{\partial F}{\partial q} \cdot \frac{\partial q}{\partial x} = F_q \cdot q_x$$

$$\Phi_x = \begin{bmatrix} \frac{\partial \Phi}{\partial x} \end{bmatrix} = \begin{bmatrix} \frac{\partial F}{\partial q} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial q}{\partial x} \end{bmatrix} = F_q \cdot q_x$$
Example

Assume that $B \in \mathbb{R}^{m \times n}$ is a matrix that doesn’t depend on $x$, where $x \in \mathbb{R}^n$. Show that:

$$\frac{\partial (Bx)}{\partial x} = B$$
Assume that \( y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \) and consider a function \( \Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) defined as \( \Phi(y) = \begin{bmatrix} 2y_1 + y_2^2 \\ y_1 y_2 \end{bmatrix} \). Assume further that \( y \) depends on a variable \( x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \) as follows:

\[
y \triangleq y(x) \equiv \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} 2x_1 + \log_{10} x_2 + \sqrt{x_3} \\ (x_1 - x_2)^2 \end{bmatrix}
\]

It follows that \( \Phi \) depends on \( x \), implicitly through \( y \). Apply the chain rule of differentiation to find the derivative of \( \Phi \) with respect to \( x \), that is:

\[
\Phi_x \triangleq \left[ \frac{\partial \Phi}{\partial x_1} \quad \frac{\partial \Phi}{\partial x_2} \quad \frac{\partial \Phi}{\partial x_3} \right] = ?
\]

What is the dimension of the result \( \Phi_x \)?
Complex Case 3
Vector Function of Vector Variables

- $F$ is a vector function of 2 vector variables $q$ and $p$:
  $$F : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{m}$$

- Both $q$ and $p$ in turn depend on a set of $k$ other variables $x = [x_1, \ldots, x_k]^T$:
  $$q : \mathbb{R}^{k} \to \mathbb{R}^{n_1} \quad q \triangleq q(x_1, \ldots, x_k)$$
  $$p : \mathbb{R}^{k} \to \mathbb{R}^{n_2} \quad p \triangleq p(x_1, \ldots, x_k)$$

- A new function $\Phi(x)$ is defined as:
  $$\Phi : \mathbb{R}^{k} \to \mathbb{R}^{m} \quad \Phi(x) \triangleq F(q(x), p(x))$$

- Example: a force (which is a vector quantity), depends on the generalized positions and velocities
Question: how do you compute $\Phi_x$?

Using our notation:

$$\Phi(x) = F(q(x), p(x)) \quad \Rightarrow \quad \Phi_x = ?$$

Chain Rule:

$$\Phi_x \equiv \frac{\partial \Phi}{\partial x} = \frac{\partial F}{\partial q} \cdot \frac{\partial q}{\partial x} + \frac{\partial F}{\partial p} \cdot \frac{\partial p}{\partial x} \equiv F_q \cdot q_x + F_p \cdot p_x$$

$$\Phi_x = \begin{bmatrix} F_q \cdot q_x + F_p \cdot p_x \end{bmatrix}_{m \times k}$$
Example

Assume that $q = q(x) \in \mathbb{R}^n$ and $p = p(x) \in \mathbb{R}^n$. Show that:

$$\frac{\partial (q^T p)}{\partial x} = q^T p_x + p^T q_x$$
Complex Case 4
Time Derivatives

- In the previous slides we talked about functions $f$ depending on $q$, where $q$ in turn depends on another variable $x$.

- The most common scenario in ME451 is when the variable $x$ is actually time, $t$.
  - You have a function that depends on the generalized coordinates $q$, and in turn the generalized coordinates are functions of time (they change in time, since we are talking about kinematics/dynamics here...)

- Case 1: scalar function that depends on an array of $m$ time-dependent generalized coordinates:
  $$\phi : \mathbb{R} \rightarrow \mathbb{R}, \quad \phi \triangleq \phi(q(t))$$

- Case 2: vector function (of dimension $n$) that depends on an array of $m$ time-dependent generalized coordinates:
  $$\Phi : \mathbb{R} \rightarrow \mathbb{R}^n, \quad \Phi \triangleq \Phi(q(t))$$
Chain Rule
Time Derivatives

- Question: what are the time derivatives of $\Phi$ and $\Phi$?

- Applying the chain rule of differentiation, the results in both cases can be written formally in the exact same way, except the dimension of the result will be different

- Case 1: scalar function

$$\dot{\phi} \triangleq \frac{d\phi}{dt} = \frac{d\phi(q(t))}{dt} = \frac{\partial\phi}{\partial q} \cdot \frac{dq}{dt} = \phi_q \dot{q}, \quad \dot{\phi} \in \mathbb{R}$$

- Case 2: vector function

$$\dot{\Phi} \triangleq \frac{d\Phi}{dt} = \frac{d\Phi(q(t))}{dt} = \frac{\partial\Phi}{\partial q} \cdot \frac{dq}{dt} = \Phi_q \dot{q}, \quad \dot{\Phi} \in \mathbb{R}^n$$
Example
Time Derivatives

Assume $q \in \mathbb{R}^3$ is an array of generalized coordinates:

$$q = \begin{bmatrix} x(t) \\ y(t) \\ \theta(t) \end{bmatrix}$$

- Find the time derivative of the scalar function $\phi(q(t)) = 3x(t) + 2L \sin \theta(t)$
- Find the time derivative of the vector function

$$\Phi = \begin{bmatrix} 3x(t) + 2L \sin \theta(t) \\ y(t) - 2L \cos \theta(t) \end{bmatrix}$$
Summary of Useful Formulas

\[ \frac{\partial}{\partial q} (g^T h) = g^T h_q + h^T g_q \]

\[ \frac{\partial}{\partial q} (Bq) = B \]

\[ \frac{\partial}{\partial p} (p^T B q) = q^T B^T \]

\[ \frac{d}{dt} (p^T B q) = q^T B^T \dot{p} + p^T B \dot{q} \]

Assumptions:

- \( g = g(q), \ h = h(q) \)
- \( B \) is a constant matrix
- \( q \) does not depend on \( p \)
- Vector and matrix dimensions are such that all operations are possible.