ME451
Kinematics and Dynamics of Machine Systems

Introduction

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Dan Negrut
University of Wisconsin-Madison

- Quote of the day: "I am so clever that sometimes I don't understand a single word of what I am saying." -Oscar Wilde

Before we get started…

- Last time
  - Discussed the concept of geometric vector (GV) – hard to manipulate
  - Introduced the concept of reference frame to describe GVs
  - GVs in reference frames are represented through a 2D algebraic vector (array of two numbers)

- Today
  - Wrap up review of linear algebra
  - Discuss change of reference frame
  - Discuss time derivatives and partial derivatives

- Reminder – HW assigned last time: 2.2.5, 2.2.8, 2.2.10
  - Due on Th, at 9:30 am

Matrix-Matrix Multiplication

- Definition

$$\mathbf{A} = [a_{ij}], \quad \mathbf{A} \in \mathbb{R}^{m \times n}$$
$$\mathbf{B} = [b_{ij}], \quad \mathbf{B} \in \mathbb{R}^{n \times p}$$
$$\mathbf{D} = \mathbf{AB} = [d_{ij}], \quad \mathbf{D} \in \mathbb{R}^{m \times p}$$
$$d_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

- Note an important prerequisite: the number of columns of $\mathbf{A}$ must be equal to the number of rows of $\mathbf{B}$
- Matrix multiplication is **not** commutative
- Associativity property: $(\mathbf{AB})\,\mathbf{C} = \mathbf{A}\,(\mathbf{BC})$
- Distributivity property: $(\mathbf{A} + \mathbf{B})\,\mathbf{C} = \mathbf{AC} + \mathbf{BC}$

Matrix-Vector Multiplication

- Definition $\mathbf{w} = \mathbf{A}\mathbf{v}$ for $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{v} \in \mathbb{R}^{n}$, $\mathbf{w} \in \mathbb{R}^{m}$

$$w_i = \sum_{j=1}^{n} a_{ij} v_j \ , \ i = 1, \ldots, m$$

- A *column-wise* perspective:

$$\mathbf{A}\mathbf{v} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \sum_{j=1}^{n} v_j \mathbf{a}_j$$

- A *row-size* perspective: $\mathbf{A}\mathbf{v} = \begin{bmatrix} \mathbf{d}_1^T \\ \mathbf{d}_2^T \\ \vdots \\ \mathbf{d}_m^T \end{bmatrix} \mathbf{v} = \begin{bmatrix} \mathbf{d}_1^T \mathbf{v} \\ \mathbf{d}_2^T \mathbf{v} \\ \vdots \\ \mathbf{d}_m^T \mathbf{v} \end{bmatrix}$

More Matrix Definitions

Transpose: The transpose of $\mathbf{A} \in \mathbb{R}^{m \times n}$, is the matrix $\mathbf{B} \triangleq \mathbf{A}^T$, $\mathbf{B} \in \mathbb{R}^{n \times m}$ which satisfies

$$b_{ij} = a_{ji}$$

Symmetric: A square matrix $\mathbf{A}$ for which $\mathbf{A} = \mathbf{A}^T$

Skew-symmetric: A square matrix $\mathbf{A}$ for which $\mathbf{A} = -\mathbf{A}^T$

Singular: A square matrix whose determinant is zero:

$$\det(\mathbf{A}) = 0$$

Inverse: The inverse of a square, non-singular matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is the matrix $\mathbf{B} \triangleq \mathbf{A}^{-1} \in \mathbb{R}^{n \times n}$ which satisfies

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_n$$

Orthogonal Matrices

Definition: An *orthogonal* matrix is a square $n \times n$ real matrix whose columns and rows are orthogonal unit vectors. This implies

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q}^T = \mathbf{I}_n$$

Important property: The inverse of an *orthogonal* matrix is its transposed

$$\mathbf{Q}^{-1} = \mathbf{Q}^T$$

Another important property: Orthogonal matrices preserve the dot product of vectors

$$(\mathbf{Q}\mathbf{a})^T(\mathbf{Q}\mathbf{b}) = \mathbf{a}^T\mathbf{b}$$

Some authors make a distinction between *orthogonal* matrices whose columns are orthogonal vectors (but not necessarily unit vectors) and which therefore satisy $\mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q}^T =$ diagonal and *orthonormal* matrices which are defined as above and satisfy $\mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q}^T = \mathbf{I}$.

Exercise

Prove that the following matrix

$$\mathbf{A} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

is *orthogonal.*

Linear Independence of Vectors

- A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$, where $\mathbf{v}_i \in \mathbb{R}^n$ is *linearly independent* if and only if

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_m\mathbf{v}_m = \mathbf{0}_n \quad \Leftrightarrow \quad \alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$$

- If the vectors are not linearly independent, they are called *linearly dependent*

- Example:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ -3 \\ -6 \end{bmatrix}$$

Since $\mathbf{v}_1 - 2\mathbf{v}_2 - \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, the three vectors are *linearly dependent.*

[handout]

Matrix Rank

- Row rank of a matrix
  - Largest number of rows of the matrix that are linearly independent
  - A matrix is said to have full row rank if the rank of the matrix is equal to the number of rows of that matrix

- Column rank of a matrix
  - Largest number of columns of the matrix that are linearly independent

- NOTE: for each matrix, the row rank and column rank are the same
  - This number is simply called the rank of the matrix
  - It follows that

$$rank\left(\mathbf{A}\right) = rank\left(\mathbf{A}^T\right)$$

Matrix Rank

Example

- What is the row rank of the following matrix?

$$\mathbf{J} = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 4 & -2 & -2 & 1 \\ 0 & -4 & 0 & 1 \end{bmatrix}$$

- What is the column rank of **J**?

Singular vs. Nonsingular Matrices

- Let **A** be a square matrix of dimension *n*. The following are equivalent:
  - $\mathbf{Ax} = \mathbf{b}$ has a unique solution for any $\mathbf{b} \in \mathbb{R}^n$;
  - $\mathbf{Ax} = \mathbf{b}$ has a solution for any $\mathbf{b} \in \mathbb{R}^n$;
  - $\mathbf{Ax} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}_n$;
  - $\mathbf{A}^{-1}$ exists;
  - $\det(\mathbf{A}) \neq 0$;
  - $rank(\mathbf{A}) = n$.

Other Useful Formulas

- If **A** and **B** are invertible, their product is invertible and

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

- Also,

$$\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$$

- For any two matrices **A** and **B** that can be multiplied

$$(\mathbf{AB})^{T} = \mathbf{B}^{T}\mathbf{A}^{T}$$

- For any two square matrices **A** and **B** of the same dimension,

$$\det(\mathbf{AB}) = \det(\mathbf{A}) \cdot \det(\mathbf{B})$$

(these are all pretty straightforward to prove, maybe except the last one)

2.4

TRANSFORMATION OF COORDINATES

Vectors and Reference Frames (1)

- Recall that an algebraic vector is just a representation of a geometric vector in a particular reference frame (RF)

$$\vec{\mathbf{s}} = s_{x'}\vec{\mathbf{i}'} + s_{y'}\vec{\mathbf{j}'} \quad \rightarrow \quad (s_{x'}, s_{y'}) \quad \rightarrow \quad \mathbf{s}' = \begin{bmatrix} s_{x'} \\ s_{y'} \end{bmatrix}$$

- Question: What if I now want to represent the same geometric vector in a different RF?

$$\vec{\mathbf{s}} = s_{x'}\vec{\mathbf{i}'} + s_{y'}\vec{\mathbf{j}'} \quad \rightarrow \quad \vec{\mathbf{s}} = s_x\vec{\mathbf{i}} + s_y\vec{\mathbf{j}}$$

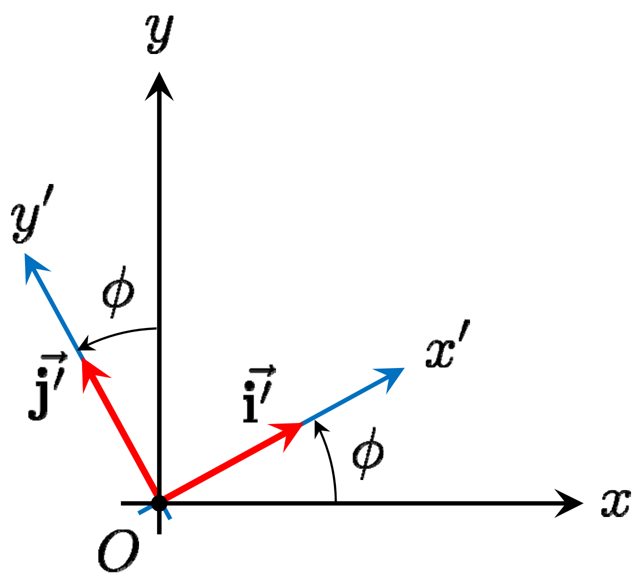
Vectors and Reference Frames (2)

- Transforming the representation of a vector from one RF to a different RF that is rotated by an angle is done through (left) multiplication by a so-called "rotation matrix" **A**(ϕ):

$$\mathbf{s} = \mathbf{A}(\phi) \cdot \mathbf{s}'$$

- Notes
  - We transform the vector's representation and not the vector itself
    - What changes is the RF used to represent the vector
  - A rotation matrix **A** is also called *"orientation matrix"*
  - Sometime the dependence of **A** on the rotation angle ϕ is dropped for brevity (yet **A** is always expressed based on a rotation angle)

The Rotation Matrix



$$\mathbf{A}(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

- Rotation matrices are orthogonal:

$$\mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T = \mathbf{I}_{2\times 2}$$

- Geometric interpretation of a rotation matrix:

$$\vec{\mathbf{i}}'\Big|_{Oxy} = \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix} \qquad \vec{\mathbf{j}}'\Big|_{Oxy} = \begin{bmatrix} -\sin\phi \\ \cos\phi \end{bmatrix} \quad \Rightarrow \quad \boxed{\mathbf{A} = \begin{bmatrix} \vec{\mathbf{i}}'\big|_{Oxy} & \vec{\mathbf{j}}'\big|_{Oxy} \end{bmatrix}}$$

Big deal matrix

Important Relation

- Expressing a vector given in one reference frame (local) in a different reference frame (global):

$$\begin{bmatrix} s_x \\ s_y \end{bmatrix} \equiv \mathbf{s} = \mathbf{A}\mathbf{s}' = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} s_{x'} \\ s_{y'} \end{bmatrix}$$

This is also called a *change of base*.

- Since the rotation matrix is orthogonal, we have

$$\mathbf{s}' = \mathbf{A}^T\mathbf{s}$$

- More acronyms:
  - LRF: local reference frame ($O'x'y'$)
  - GRF: global reference frame (Oxy)

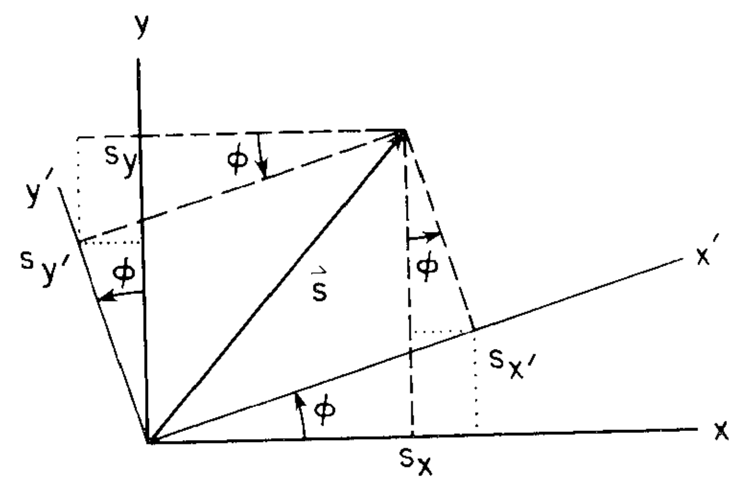


Figure 2.4.2 Vector $\vec{s}$ in two frames.