ME451
Kinematics and Dynamics of Machine Systems

Introduction
September 9, 2014

Quote of the day: "I am so clever that sometimes I don't understand a single word of what I am saying." -Oscar Wilde
Before we get started…

- Last time
  - Discussed the concept of geometric vector (GV) – hard to manipulate
  - Introduced the concept of reference frame to describe GVs
  - GVs in reference frames are represented through a 2D algebraic vector (array of two numbers)

- Today
  - Wrap up review of linear algebra
  - Discuss change of reference frame
  - Discuss time derivatives and partial derivatives

- Reminder – HW assigned last time: 2.2.5, 2.2.8, 2.2.10
  - Due on Th, at 9:30 am
Matrix-Matrix Multiplication

- Definition

\[
A = [a_{ij}], \quad A \in \mathbb{R}^{m \times n} \\
B = [b_{ij}], \quad B \in \mathbb{R}^{n \times p} \\
D = AB = [d_{ij}], \quad D \in \mathbb{R}^{m \times p} \\
d_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}
\]

- Note an important prerequisite: the number of columns of A must be equal to the number of rows of B

- Matrix multiplication is not commutative

- Associativity property: \((AB) C = A (BC)\)

- Distributivity property: \((A + B) C = AC + BC\)
Matrix-Vector Multiplication

• Definition $\mathbf{w} = \mathbf{A}\mathbf{v}$ for $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{v} \in \mathbb{R}^n$, $\mathbf{w} \in \mathbb{R}^m$

$$w_i = \sum_{j=1}^{n} a_{ij} v_j, \ i = 1, \ldots, m$$

• A column-wise perspective:

$$\mathbf{A}\mathbf{v} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \sum_{j=1}^{n} v_j a_j$$

• A row-size perspective: $\mathbf{A}\mathbf{v} = \begin{bmatrix} \mathbf{d}_1^T \\ \mathbf{d}_2^T \\ \vdots \\ \mathbf{d}_m^T \end{bmatrix} \mathbf{v} = \begin{bmatrix} \mathbf{d}_1^T \mathbf{v} \\ \mathbf{d}_2^T \mathbf{v} \\ \vdots \\ \mathbf{d}_m^T \mathbf{v} \end{bmatrix}$
More Matrix Definitions

**Transpose:** The transpose of $A \in \mathbb{R}^{m \times n}$, is the matrix $B \triangleq A^T$, $B \in \mathbb{R}^{n \times m}$ which satisfies

$$b_{ij} = a_{ji}$$

**Symmetric:** A square matrix $A$ for which $A = A^T$

**Skew-symmetric:** A square matrix $A$ for which $A = -A^T$

**Singular:** A square matrix whose determinant is zero:

$$\det(A) = 0$$

**Inverse:** The inverse of a square, non-singular matrix $A \in \mathbb{R}^{n \times n}$ is the matrix $B \triangleq A^{-1} \in \mathbb{R}^{n \times n}$ which satisfies

$$A^{-1}A = AA^{-1} = I_n$$
**Orthogonal Matrices**

**Definition:** An *orthogonal* matrix is a square $n \times n$ real matrix whose columns and rows are *orthogonal unit* vectors. This implies

$$Q^T Q = QQ^T = I_n$$

**Important property:** The inverse of an *orthogonal* matrix is its transposed

$$Q^{-1} = Q^T$$

**Another important property:** Orthogonal matrices preserve the dot product of vectors

$$(Qa)^T (Qb) = a^T b$$

Some authors make a distinction between *orthogonal* matrices whose columns are orthogonal vectors (but not necessarily unit vectors) and which therefore satisfy $Q^T Q = QQ^T = \text{diagonal}$ and *orthonormal* matrices which are defined as above and satisfy $Q^T Q = QQ^T = I$. 
Exercise

Prove that the following matrix

\[ A = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \]

is orthogonal.
Linear Independence of Vectors

• A set of vectors \( \{v_1, \ldots, v_m\} \), where \( v_i \in \mathbb{R}^n \) is \textit{linearly independent} if and only if

\[
\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m = 0_n \iff \alpha_1 = \alpha_2 = \cdots = \alpha_m = 0
\]

• If the vectors are not linearly independent, they are called \textit{linearly dependent}

• Example:

\[
v_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ -3 \\ -6 \end{bmatrix}
\]

Since \( v_1 - 2v_2 - v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \), the three vectors are \textit{linearly dependent}.
Matrix Rank

- **Row rank of a matrix**
  - Largest number of rows of the matrix that are linearly independent
  - A matrix is said to have full row rank if the rank of the matrix is equal to the number of rows of that matrix

- **Column rank of a matrix**
  - Largest number of columns of the matrix that are linearly independent

- **NOTE**: for each matrix, the row rank and column rank are the same
  - This number is simply called the rank of the matrix
  - It follows that

\[
\text{rank}(A) = \text{rank}(A^T)
\]
Matrix Rank
Example

What is the row rank of the following matrix?

\[ J = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 4 & -2 & -2 & 1 \\ 0 & -4 & 0 & 1 \end{bmatrix} \]

What is the column rank of \( J \)?
Let \( A \) be a square matrix of dimension \( n \). The following are equivalent:

- \( Ax = b \) has a unique solution for any \( b \in \mathbb{R}^n \);
- \( Ax = b \) has a solution for any \( b \in \mathbb{R}^n \);
- \( Ax = 0 \) implies \( x = 0_n \);
- \( A^{-1} \) exists;
- \( \det(A) \neq 0 \);
- \( \text{rank}(A) = n \).
Other Useful Formulas

- If $A$ and $B$ are invertible, their product is invertible and
  \[(AB)^{-1} = B^{-1}A^{-1}\]

- Also,
  \[(A^{-1})^{-1} = A\]

- For any two matrices $A$ and $B$ that can be multiplied
  \[(AB)^T = B^TA^T\]

- For any two square matrices $A$ and $B$ of the same dimension,
  \[\text{det} (AB) = \text{det} (A) \cdot \text{det} (B)\]

(these are all pretty straightforward to prove, maybe except the last one)
2.4

TRANSFORMATION OF COORDINATES
Vectors and Reference Frames (1)

- Recall that an algebraic vector is just a representation of a geometric vector in a particular reference frame (RF)

\[ \vec{s} = s_x \vec{i} + s_y \vec{j} \quad \rightarrow \quad (s_x', s_y') \quad \rightarrow \quad s' = \begin{bmatrix} s_x' \\ s_y' \end{bmatrix} \]

- Question: What if I now want to represent the same geometric vector in a different RF?

\[ \vec{s} = s_x \vec{i} + s_y \vec{j} \quad \rightarrow \quad \vec{s} = s_x \vec{i} + s_y \vec{j} \]
Vectors and Reference Frames (2)

- Transforming the representation of a vector from one RF to a different RF that is rotated by an angle is done through (left) multiplication by a so-called “rotation matrix” $A(\phi)$:

$$s = A(\phi) \cdot s'$$

- Notes
  - We transform the vector’s representation and not the vector itself
    - What changes is the RF used to represent the vector
  - A rotation matrix $A$ is also called “orientation matrix”
  - Sometime the dependence of $A$ on the rotation angle $\phi$ is dropped for brevity (yet $A$ is always expressed based on a rotation angle)
The Rotation Matrix

Rotation matrices are orthogonal:

\[
A(\phi) = \begin{bmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{bmatrix}
\]

- Rotation matrices are **orthogonal**:
  \[
  A^T A = A A^T = I_{2 \times 2}
  \]

- Geometric interpretation of a rotation matrix:

  \[
  i'_{Oxy} = \begin{bmatrix}
  \cos \phi \\
  \sin \phi
  \end{bmatrix}
  \]

  \[
  j'_{Oxy} = \begin{bmatrix}
  -\sin \phi \\
  \cos \phi
  \end{bmatrix}
  \]

  \[
  \Rightarrow \quad A = \begin{bmatrix}
  i'_{Oxy} & j'_{Oxy}
  \end{bmatrix}
  \]

Big deal matrix
Important Relation

- Expressing a vector given in one reference frame (local) in a different reference frame (global):

\[
\begin{bmatrix}
    s_x \\
    s_y
\end{bmatrix} \equiv s = \mathbf{A}s' = \begin{bmatrix}
    \cos \phi & -\sin \phi \\
    \sin \phi & \cos \phi
\end{bmatrix} \begin{bmatrix}
    s_{x'} \\
    s_{y'}
\end{bmatrix}
\]

This is also called a change of base.

- Since the rotation matrix is orthogonal, we have

\[
s' = \mathbf{A}^T s
\]

- More acronyms:
  - LRF: local reference frame ($O'x'y'$)
  - GRF: global reference frame ($Oxy$)

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*Figure 2.4.2* Vector $\mathbf{s}$ in two frames.