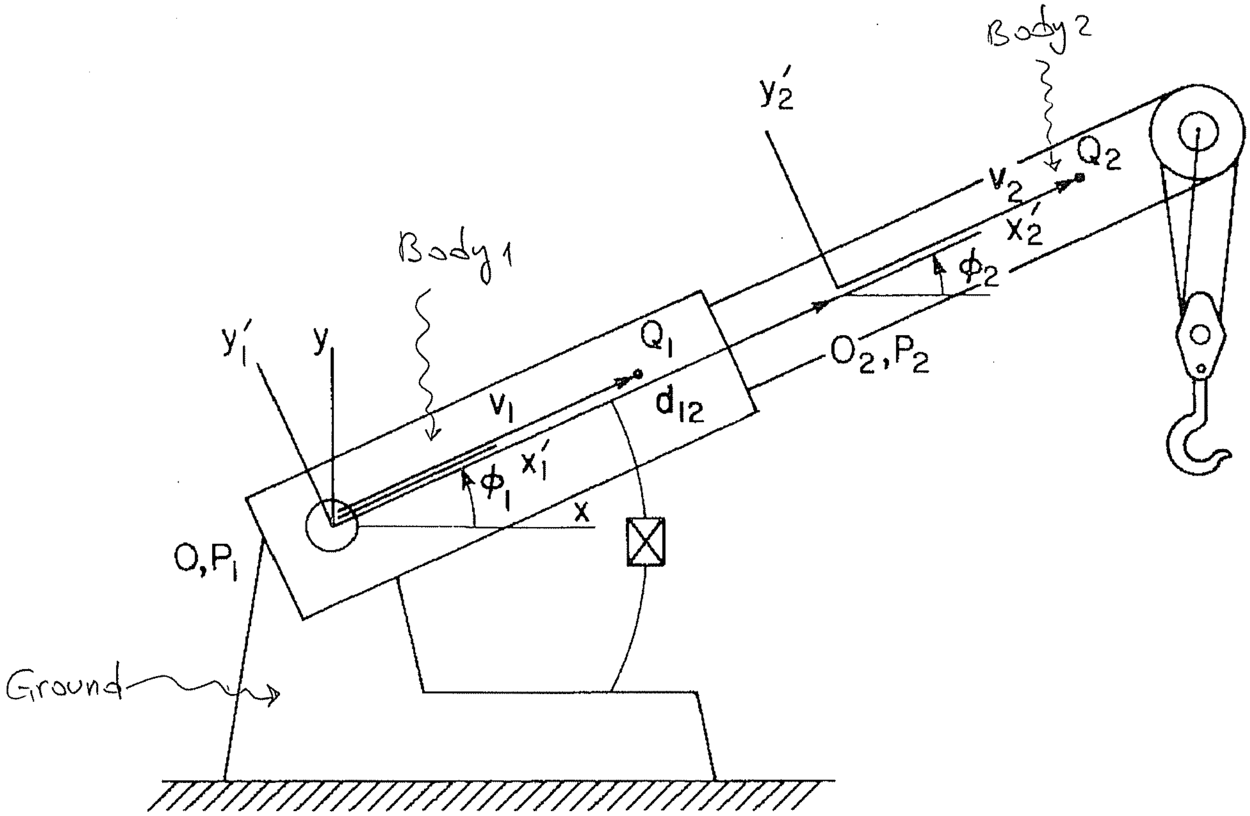

Figure 3.5.10 Wrecker boom with a translational-distance driver.

Generalized coordinates:

$$q_1 = \begin{bmatrix} x_1 \\ y_1 \\ \phi_1 \end{bmatrix} \qquad q_2 = \begin{bmatrix} x_2 \\ y_2 \\ \phi_2 \end{bmatrix} \qquad q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ \phi_1 \\ r_2 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ \phi_1 \\ x_2 \\ y_2 \\ \phi_2 \end{bmatrix}$$

$\boxed{\text{STEP A}}$: Identify joints and drivers

$\boxed{1}$ Absolute position constraint of point P_1 on body 1. (x & y components)

$$C_1 = 0 \qquad C_2 = 0 \qquad s_1'^P = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ (see Eq. 3.2.3 & 3.2.4)

$\boxed{2}$ Rotational driver between Body 1 and Ground.

This is an absolute driver constraint, use Eq 3.5.3.

$$C_3(t) = 0.025\,t$$

$\boxed{3}$ Translational joint between Body 1 & 2.

v_1' - defined by P_1 and Q_1: $s_1'^P = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $s_1'^Q = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

v_2' - defined by P_2 and Q_2 $s_2'^P = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $s_2'^Q = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$$v_1' = s_1'^Q - s_1'^P = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad v_2' = s_2'^Q - s_2'^P = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(see Eq. 3.3.13)

$\boxed{4}$ Translational distance driver, between Body 1 & 2.

In our case $\|v_1'\| = \|v_2'\| = 1$ (the vectors have norm 1).

v_1' & v_2' are defined like in $\boxed{3}$ above.

(see Eq. 3.5.12).

$\boxed{\text{STEP B}}$

Getting together $\phi(q,t)$.

$\phi^{\boxed{1}} \rightarrow$ use Eqs. 3.2.3 & 3.2.4.

$$\phi^{\boxed{1}}(q) = r_1^P - r_0^P = \underbrace{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + A_1 \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{s_1'^P}}_{r_1^P} - \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{r_0^P} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore, $\phi^{\boxed{1}}(q) = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$.

$\phi^{\boxed{2}} \rightarrow$ use Eq. 3.5.3

$$\phi^{(2)}(q,t) = \Phi_1 - C_3(t)$$

$$= \phi_1 - 0.025t = 0.$$

$\longleftrightarrow$

$\phi^{(3)}(q) \rightarrow$ use Eq. 3.3.13

$$\phi^{(3)}(q) = \begin{bmatrix} v_1'^T B_1^T (r_2 - r_1) - v_1'^T B_{12} s_2'^P - v_1' R^T s_1'^P \\ - v_1'^T B_{12} v_2' \end{bmatrix}$$

$$B_1 v_1' = \begin{bmatrix} -s\phi_1 & -c\phi_1 \\ c\phi_1 & -s\phi_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -s\phi_1 \\ c\phi_1 \end{bmatrix} \Rightarrow v_1'^T B_1^T = [-s\phi_1 \quad c\phi_1]$$

$$B_{12} v_2' = \begin{bmatrix} -s(\phi_2 - \phi_1) & -c(\phi_2 - \phi_1) \\ c(\phi_2 - \phi_1) & -s(\phi_2 - \phi_1) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -s(\phi_2 - \phi_1) \\ c(\phi_2 - \phi_1) \end{bmatrix}$$

$$R^T s_1'^P = B_{12} s_2'^P = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \& \quad v_1'^T B_1^T (r_2 - r_1) = [-s\phi_1 \quad c\phi_1] \cdot \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$$

Then,

$$\phi^{(3)}(q) = \begin{bmatrix} -(x_2 - x_1) s\phi_1 + (y_2 - y_1) c\phi_1 \\ s(\phi_2 - \phi_1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\longleftrightarrow$

$\phi^{(4)}(p,t) \rightarrow$ use 3.5.12. $\quad i = 1 \quad j = 2$

$$v_1' = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad s_2'^P = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad s_1'^P = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\phi^{(4)}(q,t) = v_1'^T A_1^T (r_2 - r_1) + v_1'^T \cancelto{0}{A_{12} s_2'^P} - \cancelto{0}{v_1'^T s_1'^P} - c(t) = 0$$

Then

$$\phi^{\boxed{4}}(p,t) = \begin{bmatrix} x_2-x_1 & y_2-y_1 \end{bmatrix} \cdot \begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \overset{-c(t)}{=} \begin{bmatrix} x_2-x_1 & y_2-y_1 \end{bmatrix} \begin{bmatrix} c\phi_1 \\ s\phi_1 \end{bmatrix} - c(t)$$

$$= (x_2-x_1)c\phi_1 + (y_2-y_1)s\phi_1 - c(t) = 0$$

The overall set of constraints is:

$$\phi(q,t) = \begin{bmatrix} x_1 \\ y_1 \\ \phi_1 - 0.025t \\ -(x_2-x_1)s\phi_1 + (y_2-y_1)c\phi_1 \\ s(\phi_2-\phi_1) \\ (x_2-x_1)c\phi_1 + (y_2-y_1)s\phi_1 - 0.1t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This represents a system of nonlinear equations. Given a time t_k, one can solve this system for q_k.

Next we have to compute the Jacobian ϕ_q. We can do this in two ways. First by looking at $\phi(q,t)$ above I can compute ϕ_q as follows:

$$\phi_q(q,t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ s\phi_1 & -c\phi_1 & -(x_2-x_1)c\phi_1 - (y_2-y_1)s\phi_1 & -s\phi_1 & c\phi_1 & 0 \\ 0 & 0 & -c(\phi_2-\phi_1) & 0 & 0 & c(\phi_2-\phi_1) \\ -c\phi_1 & -s\phi_1 & -(x_2-x_1)s\phi_1 + (y_2-y_1)c\phi_1 & c\phi_1 & s\phi_1 & 0 \end{bmatrix}$$

The second approach is to use the expression of the partial derivatives provided in the book.

$\Phi_q^{[1]} \rightarrow$ use Eq 3.2.5

$$\Phi_{q_1}^{[1]} = [\, I_{2\times 2} \,|\, B_1 s_1'^P \,] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right]$$

$$\underbrace{}_{I} \quad \underbrace{}_{B_1 s_1'^P}$$

Where does $\Phi_{q_1}^{[1]}$ go into the big Φ_q?

In $\Phi(q,t)$, $\Phi^{[1]}$ occupies rows 1 & 2.

$\Phi_{q_1}^{[1]}$ is the partial with respect to entries 1, 2, and 3 of q.

Therefore $\Phi_{q_1}^{[1]}$ goes in rows (1 & 2), columns (1 & 2 & 3).

$\Phi_q^{[2]} \rightarrow$ use Eq. 3.2.7

$$\Phi_{q_1}^{[2]} = [\, 0 \quad 0 \quad 1 \,].$$

In $\Phi(q,t)$, $\Phi^{[2]}$ occupies row 3.

In q, q_1 occupies entries 1, 2, 3.

Then $\Phi_{q_1}^{[2]}$ goes into Φ_q in row 3, columns (1 & 2 & 3).

$\Phi_q^{[3]} \rightarrow$ use Eq. 3.3.14.

$$\Phi_{q_1}^{[3]} = \begin{bmatrix} -v_1'^T B_1^T & -v_1'^T A_1^T (r_2 - r_1) - v_1'^T A_{12} \cancel{s_2'^P}^{\,0} \\ 0 & -v_1'^T A_{12} v_2' \end{bmatrix}$$

$$-v_1'^T A_1^T (r_2 - r_1) = -\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} c\phi_1 & s\phi_1 \\ -s\phi_1 & c\phi_1 \end{bmatrix} (r_2 - r_1)$$

$$= \begin{bmatrix} -c\phi_1 & -s\phi_1 \end{bmatrix} \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} = -(x_2 - x_1)c\phi_1 - (y_2 - y_1)s\phi_1 .$$

Next,

$$-v_1'^T A_{12} v_2' = -\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} c(\phi_2 - \phi_1) & -s(\phi_2 - \phi_1) \\ s(\phi_2 - \phi_1) & c(\phi_2 - \phi_1) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} c(\phi_2 - \phi_1) \\ s(\phi_2 - \phi_1) \end{bmatrix}$$

$$= -c(\phi_2 - \phi_1)$$

Then,

$$\phi_{q_1}^{\boxed{3}} = \begin{bmatrix} s\phi_1 & -c\phi_1 & -(x_2 - x_1)c\phi_1 - (y_2 - y_1)s\phi_1 \\ 0 & 0 & -c(\phi_2 - \phi_1) \end{bmatrix}$$

In $\phi(q,t)$, $\phi^{\boxed{3}}$ occupies rows 4 & 5.

In q, q_1 occupies rows 1, 2, 3.

Then $\phi_{q_1}^{\boxed{3}}$ goes into ϕ_q in rows (4&5), columns (1 & 2 & 3).

Next we compute $\phi_{q_2}^{\boxed{3}}$. (see Eq. 3.3.14)

$$\phi_{q_2}^{\boxed{3}} = \left[\begin{array}{c:c} v_1'^T B_1^T & v_1'^T \cancel{A_{12}} s_2'^P \overset{0}{} \\ 0_{1\times 2} & v_1'^T A_{12} v_2' \end{array}\right] = \left[\begin{array}{cc:c} -s\phi_1 & c\phi_1 & 0 \\ 0 & 0 & c(\phi_2 - \phi_1) \end{array}\right]$$

In $\phi(q,t)$, $\phi^{\boxed{3}}$ occupies rows 4 & 5.

In q, q_2 occupies rows 4, 5, 6.

Then $\phi_{q_2}^{\boxed{3}}$ goes into ϕ_q in rows (4&5), columns (4 & 5 & 6).

To compute $\Phi_{q_1}^{(4)}$ use Eq 3.5.13.

$$\Phi_{q_1}^{\boxed{4}} = \left[-v_1'^T A_1^T \qquad v_1'^T B_1^T (r_2 - r_1) \right]$$

$$A_1 v_1' = \begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c\phi_1 \\ s\phi_1 \end{bmatrix}.$$

$$v_1'^T B_1^T (r_2 - r_1) = \begin{bmatrix} -s\phi_1 & c\phi_1 \end{bmatrix} \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} = -(x_2 - x_1) s\phi_1 + (y_2 - y_1) c\phi_1$$

Then,

$$\Phi_{q_1}^{\boxed{4}} = \left[-c\phi_1 \quad -s\phi_1 \qquad -(x_2 - x_1) s\phi_1 + (y_2 - y_1) c\phi_1 \right].$$

In $\Phi(q,t)$, $\Phi^{(4)}$ occupies row 6.

In q, q_1 occupies rows 1, 2, 3.

Then $\Phi_{q_1}^{\boxed{4}}$ goes in Φ_q in row 6, columns (1 & 2 & 3).

To compute $\Phi_{q_2}^{(4)}$ use Eq. 3.5.13:

$$\Phi_{q_2}^{(4)} = \left[v_1'^T A_1^T \;\vdots\; 0 \right] = \left[c\phi_1 \quad s\phi_1 \;\vdots\; 0 \right].$$

In $\Phi(q,t)$, $\Phi^{(4)}$ occupies row 6.

In q, q_2 occupies rows 4, 5, 6.

Then $\Phi_{q_2}^{\boxed{4}}$ goes in Φ_q in row 6, columns (4 & 5 & 6).

At this point we have all the information required to assemble Φ_q (see next page).

$$\Phi_q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ s\phi_1 & -c\phi_1 & -(x_2-x_1)c\phi_1 - (y_2-y_1)s\phi_1 & -s\phi_1 & c\phi_1 & 0 \\ 0 & 0 & -c(\phi_2-\phi_1) & 0 & 0 & c(\phi_2-\phi_1) \\ -c\phi_1 & -s\phi_1 & -(x_2-x_1)s\phi_1 + (y_2-y_1)c\phi_1 & c\phi_1 & s\phi_1 & 0 \end{bmatrix}$$

Note that this Φ_q is identical to the one on page 4, which was determined by direct differentiation.

Next, to solve for velocity one needs $\nu = -\Phi_t$.

$$-\Phi_t = \begin{bmatrix} 0 \\ 0 \\ 0.025 \\ 0 \\ 0 \\ 0.1 \end{bmatrix}$$

The velocities are found after finding the position q_k as

$$\Phi_q \cdot \dot{q}_k = \nu$$

Finally, one has to solve for the accelerations, and for this we need γ.

For $\Phi^{(1)} \rightarrow$ use Eqs at top of page 60.

$$x_1'^{P} = y_1'^{P} = 0 \quad \Rightarrow \quad \gamma = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For $\phi^{(2)} \rightarrow$ use 3.5.3

$$\gamma = \frac{\partial^2 C_3(t)}{\partial t^2} = 0.$$

For $\phi^{(3)} \rightarrow$ use Eq. on page 68 (note that a square bracket is missing in that equation)

$$\gamma^{t(1,2)} = \begin{bmatrix} v_1'^{T}\left[-B_{12} s_2'^{P} (\dot{\phi}_2 - \dot{\phi}_1)^2 + B_1^T (r_2 - r_1)\dot{\phi}_1^2 + 2A_1^T(\dot{r}_2 - \dot{r}_1)\dot{\phi}_1\right] \\ 0 \end{bmatrix}$$

$$s_2'^{P} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1'^{T} B_1^T = [1 \quad 0] \begin{bmatrix} -s\phi_1 & c\phi_1 \\ -c\phi_1 & -s\phi_1 \end{bmatrix} = [-s\phi_1 \quad c\phi_1]$$

$$v_1'^{T} A_1^T = [1 \quad 0] \begin{bmatrix} c\phi_1 & s\phi_1 \\ -s\phi_1 & c\phi_1 \end{bmatrix} = [c\phi_1 \quad s\phi_1]$$

$$\gamma^{t(1,2)} = \begin{bmatrix} \dot{\phi}_1^2 \, [-s\phi_1 \quad c\phi_1] \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} + 2\,[c\phi_1 \quad s\phi_1] \begin{bmatrix} \dot{x}_2 - \dot{x}_1 \\ \dot{y}_2 - \dot{y}_1 \end{bmatrix} \dot{\phi}_1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \dot{\phi}_1^2 \left[-s\phi_1 (x_2 - x_1) + c\phi_1 (y_2 - y_1)\right] + 2\dot{\phi}_1 \left[c\phi_1 (\dot{x}_2 - \dot{x}_1) + s\phi_1 (\dot{y}_2 - \dot{y}_1)\right] \\ 0 \end{bmatrix}$$

For ϕ^{4},

$$\gamma^{tdd(1,2)} = -2\dot{\phi}_1 v_1'^T B_1^T (\dot{r}_2-\dot{r}_1) + v_1'^T \dot{\phi}_1^2 A_1^T (r_2-r_1) + \frac{\partial^2 c(t)}{\partial t^2}$$

$$= -2\dot{\phi}_1 \begin{bmatrix} -s\phi_1 & c\phi_1 \end{bmatrix} \begin{bmatrix} \dot{x}_2-\dot{x}_1 \\ \dot{y}_2-\dot{y}_1 \end{bmatrix} + \dot{\phi}_1^2 \begin{bmatrix} c\phi_1 & s\phi_1 \end{bmatrix} \begin{bmatrix} x_2-x_1 \\ y_2-y_1 \end{bmatrix}$$

$$= -2\dot{\phi}_1 \left[-s\phi_1(\dot{x}_2-\dot{x}_1) + c\phi_1(\dot{y}_2-\dot{y}_1) \right] + \dot{\phi}_1^2 \left[c\phi_1(x_2-x_1) + s\phi_1(y_2-y_1) \right]$$

Based on γ for ϕ^{1} through ϕ^{4}, we assemble the overall γ:

$$\gamma = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dot{\phi}_1^2 \left[-s\phi_1(x_2-x_1) + c\phi_1(y_2-y_1) \right] + 2\dot{\phi}_1 \left[c\phi_1(\dot{x}_2-\dot{x}_1) + s\phi_1(\dot{y}_2-\dot{y}_1) \right] \\ 0 \\ 2\dot{\phi}_1 \left[s\phi_1(\dot{x}_2-\dot{x}_1) - c\phi_1(\dot{y}_2-\dot{y}_1) \right] + \dot{\phi}_1^2 \left[c\phi_1(x_2-x_1) + s\phi_1(y_2-y_1) \right] \end{bmatrix}$$

We can verify the correctness of this result by taking two straight derivatives of our $\Phi(q,t)$ (see page 4 of this document):

$$\dot{\Phi}(q,t) = \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{\phi}_1 - 0.025 \\ -(\dot{x}_2-\dot{x}_1)s\phi_1 - (x_2-x_1)\dot{\phi}_1 c\phi_1 + (\dot{y}_2-\dot{y}_1)c\phi_1 - (y_2-y_1)\dot{\phi}_1 s\phi_1 \\ c(\phi_2-\phi_1)(\dot{\phi}_2-\dot{\phi}_1) \\ (\dot{x}_2-\dot{x}_1)c\phi_1 - (x_2-x_1)\dot{\phi}_1 s\phi_1 + (\dot{y}_2-\dot{y}_1)s\phi_1 + (y_2-y_1)\dot{\phi}_1 c\phi_1 - 0.1 \end{bmatrix}$$

$$\ddot{\phi}(q,t) = \begin{bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\phi}_1 \\ -(\ddot{x}_2-\ddot{x}_1)s\phi_1 - 2(\dot{x}_2-\dot{x}_1)\dot{\phi}_1 c\phi_1 - (x_2-x_1)\ddot{\phi}_1 c\phi_1 + (x_2-x_1)\dot{\phi}_1^2 s\phi_1 + (\ddot{y}_2-\ddot{y}_1)c\phi_1 - 2(\dot{y}_2-\dot{y}_1)\dot{\phi}_1 s\phi_1 - (y_2-y_1)\ddot{\phi}_1 s\phi_1 - (y_2-y_1)\dot{\phi}_1^2 c\phi_1 \\ \cancelto{0}{s(\phi_2-\phi_1)}(\dot{\phi}_2-\dot{\phi}_1)^2 + c(\phi_2-\phi_1)(\ddot{\phi}_2-\ddot{\phi}_1) \\ (\ddot{x}_2-\ddot{x}_1)c\phi_1 - 2(\dot{x}_2-\dot{x}_1)\dot{\phi}_1 s\phi_1 - (x_2-x_1)\ddot{\phi}_1 s\phi_1 - (x_2-x_1)\dot{\phi}_1^2 c\phi_1 + (\ddot{y}_2-\ddot{y}_1)s\phi_1 + 2(\dot{y}_2-\dot{y}_1)\dot{\phi}_1 c\phi_1 + (y_2-y_1)\ddot{\phi}_1 c\phi_1 - (y_2-y_1)\dot{\phi}_1^2 s\phi_1 \end{bmatrix}$$

Then, separating γ out of $\ddot{\phi}(q,t)$ one ends up with

$$\gamma = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2\dot{\phi}_1 \cdot [-(\dot{x}_2-\dot{x}_1)c\phi_1 + (\dot{y}_2-\dot{y}_1)s\phi_1] + \dot{\phi}_1^2[-(x_2-x_1)s\phi_1 + (y_2-y_1)c\phi_1] \\ 0 \\ 2\dot{\phi}_1[(\dot{x}_2-\dot{x}_1)s\phi_1 - (\dot{y}_2-\dot{y}_1)c\phi_1] + \dot{\phi}_1^2[(x_2-x_1)c\phi_1 + (y_2-y_1)s\phi_1] \end{bmatrix}$$

Note that, as expected, this is identical to γ derived at page 10 by using the γ formulas provided by the book.

Putting things in perspective, we derived all the information needed to carry out position, velocity and acceleration analysis.

We used the formulas provided in the book to find the partial derivatives required to assemble Φ_q, we used appropriate formulas to come up with ν, and finally to produce the RHS of the acceleration equation.

Note that we had to assemble pieces of information provided by the formulas based on the global index associated with each constraint and each generalized coordinate. In this context, $\phi^{(1)}$ had global index 1 & 2, $\phi^{(2)}$ global index 3, $\phi^{(3)}$ had global index 4 & 5, and $\phi^{(4)}$ had global index 6. By the same token, x_1, y_1, ϕ_1 had global index 1, 2, and 3, while the generalized coordinates associated with body 2 had global index in the overall generalized coordinate vector 4, 5, and 6.

These global indeces are important when assembling Φ, ν, γ, and Φ_q.