 Name (Print): ____________________

ME 451: Midterm Exam 1

Problem A. [**40 points**]

1. (10 points)

   Let $\mathbf{M}$ be a constant matrix of size $n \times n$ and let $\mathbf{q}$ be an array of m generalized coordinates. Consider two arrays $\mathbf{g}$ and $\mathbf{h}$, each of dimension n, that depend on the generalized coordinates $\mathbf{q}$. That is, $\mathbf{g} = \mathbf{g}(\mathbf{q})$ and $\mathbf{h} = \mathbf{h}(\mathbf{q})$. Write down the partial derivative of $\left(\mathbf{g}^T \mathbf{M} \mathbf{h}\right)$ with respect to $\mathbf{q}$. What is the dimension (size) of the resulting quantity?

2. (10 points)

   What is the significance of the *Implicit Function Theorem* in the context of Kinematic Analysis?

3. (10 points)

   What is the role played by the Jacobian matrix $\mathbf{\Phi_q}$ in Kinematic Analysis?

4. (10 points)

   How can the Newton-Raphson method fail? What is the most common cause for these problems and how can you address them?

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5. (Bonus: 10 points)

   Consider a set of three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, each of dimension n, assumed to be *linearly independent*. Show that the vectors $\mathbf{a}$, $\mathbf{a}+\mathbf{b}$, and $\mathbf{a}+\mathbf{b}+\mathbf{c}$ also form a set of *linearly independent* vectors.

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Problem B. [**60 points**]

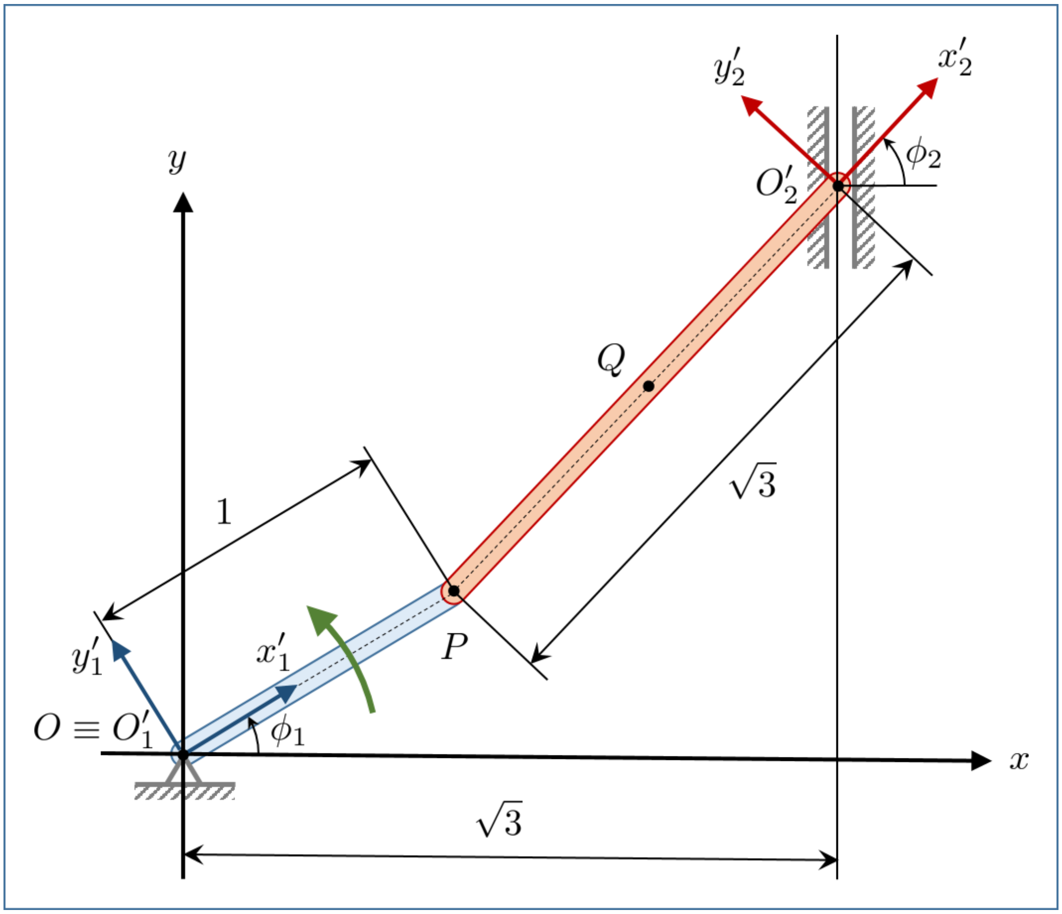

Figure 1: Two body mechanism

Consider the mechanism in Figure 1, modeled using a full set of absolute (Cartesian) coordinates

$$\mathbf{q} = \left[x_1 \, , \, y_1 \, , \, \phi_1 \, , \, x_2 \, , \, y_2 \, , \, \phi_2 \right] \, ,$$

with the local reference frames for the two bodies located as indicated in the figure. Assume further that a motion is prescribed as

$$\phi_1 = \frac{\pi}{6} + \sin(2t) \, .$$

All dimensions, as indicated in the figure are in SI units.

1. (10 points)

   Describe (in *words*) the **kinematic constraints** you would use to model this mechanism. How many (kinematic) degrees of freedom does the mechanism have?

2. (15 points)

   Using the set of generalized coordinates indicated above, specify the set of kinematic constraint equations $\mathbf{\Phi}^K(\mathbf{q}) = \mathbf{0}$ that model this mechanism.

3. (5 points)

   Specify the driver constraint equations $\mathbf{\Phi}^D(\mathbf{q}, t) = \mathbf{0}$, required to impose the desired motion. How many such constraints did you impose and why?

4. (20 points)

   Find the generalized coordinates $\mathbf{q}$ and generalized velocities $\dot{\mathbf{q}}$ at time $t = 0$. In other words, solve the *position analysis* and *velocity analysis* problems at the initial time $t = 0$.

5. (10 points)

   Consider the point Q located at the center of body 2 (assumed to be symmetric). Calculate the position and velocity of point Q at the initial configuration calculated above (at $t = 0$).

6. (Bonus: 15 points)

   At time $t = 0.01s$, carry out the first iteration of the Newton-Raphson iterative process that you would use in performing *position analysis*. As a starting point, use the values of the generalized coordinates at time $t = 0$ (that is, the initial configuration calculated above).

   Notes:

   - You can (but are not required to) use a reduced set of generalized coordinates, $\mathbf{q} = \left[\phi_1\,,\, y_2\,,\, \phi_2\right]$.
   - You do **not** have to solve for the first approximation of the generalized states $\mathbf{q}(t_1)$. Simply write down the equation that you would have to solve in order to calculate them.

7. (Bonus: 15 points)

   Is it possible for this mechanism to reach a singular configuration? Why or why not?

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ME 451: Midterm Exam 1

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1. (10 points)

   Let $\mathbf{M}$ be a constant matrix of size $n \times n$ and let $\mathbf{q}$ be an array of m generalized coordinates. Consider two arrays $\mathbf{g}$ and $\mathbf{h}$, each of dimension n, that depend on the generalized coordinates $\mathbf{q}$. That is, $\mathbf{g} = \mathbf{g}(\mathbf{q})$ and $\mathbf{h} = \mathbf{h}(\mathbf{q})$. Write down the partial derivative of $(\mathbf{g}^T\mathbf{M}\mathbf{h})$ with respect to $\mathbf{q}$. What is the dimension (size) of the resulting quantity?

   Let $F: \mathbb{R}^m \to \mathbb{R}$; $F(q) \triangleq g^T(q)\, M\, h(q)$. We therefore expect to get $F_q \in \mathbb{R}^{1\times m}$

   Using the chain rule : $F_q = \dfrac{\partial F}{\partial g} \cdot \dfrac{\partial g}{\partial q} + \dfrac{\partial F}{\partial h} \cdot \dfrac{\partial h}{\partial q}$

   But $\dfrac{\partial F}{\partial g} = \dfrac{\partial (g^T M h)}{\partial g} = \dfrac{\partial ((g^T M h)^T)}{\partial g} = \dfrac{\partial (h^T M^T g)}{\partial g} = h^T M^T$

   $\dfrac{\partial F}{\partial h} = \dfrac{\partial (g^T M h)}{\partial h} = g^T M$

   $$\Rightarrow F_q = \underbrace{\underset{1\times n}{h^T}\ \underset{n\times n}{M^T}\ \underset{n\times m}{g_q} + \underset{1\times n}{g^T}\ \underset{n\times n}{M}\ \underset{n\times m}{h_q}}_{1\times m}$$

2. (10 points)

   What is the significance of the *Implicit Function Theorem* in the context of Kinematic Analysis?

   The significance of the IFT is that it provides the conditions (Jacobian nonsingular) under which there is a unique solution to the Kinematic Position Analysis Problem. Under additional conditions (constraint equations are twice differentiable), it also guarantees that the Velocity and Acceleration Problems are well posed.
   As a consequence, IFT provides a mechanism to check for possible singular configurations (configurations where the Jacobian is singular)

3. (10 points)

   What is the role played by the Jacobian matrix $\Phi_{\mathbf{q}}$ in Kinematic Analysis?

   The Jacobian matrix is crucial in verifying if a solution to the Pos. An. exists and is unique (through the IFT). The Jacobian is the matrix in the linear Vel. and Acc. problems. Moreover, the Newton-Raphson algorithm relies on the Jacobian matrix to solve for the update vector at each of its iterations.

4. (10 points)

   How can the Newton-Raphson method fail? What is the most common cause for these problems and how can you address them?

   Different mechanism through which N-R can fail are : divergence, cycling, convergence to a different root, division by zero at extrema, etc.
   All these issues are due to an initial guess that is too far from the solution we are seeking and can therefore be prevented by providing better initial guess. The only failure mechanism that can not be addressed by a more precise initial guess is the (pathological) situation where the Jacobian is singular at the solution!

5. (Bonus: 10 points)

Consider a set of three vectors **a**, **b**, and **c**, each of dimension n, assumed to be *linearly independent*. Show that the vectors **a**, **a** + **b**, and **a** + **b** + **c** also form a set of *linearly independent* vectors.

Given: $\{a, b, c\}$ are linearly independent. This means that the only way to make $\alpha a + \beta b + \gamma c = 0$ is by setting $\alpha = \beta = \gamma = 0$.

Consider now the set of vectors

$$u \triangleq a \quad ; \quad v \triangleq a+b \quad ; \quad w \triangleq a+b+c$$

To check whether or not $\{u, v, w\}$ are linearly independent, we form the combination $\alpha' u + \beta' v + \gamma' w$ ~~and~~ for some real numbers α', β', γ' and then see what we can say about α', β', γ' if we impose that $\alpha' u + \beta' v + \gamma' w = 0$.

But this means imposing that $\alpha' a + \beta'(a+b) + \gamma'(a+b+c) = 0$

Rearranging, this means: $(\alpha' + \beta' + \gamma') a + (\beta' + \gamma') b + \gamma' c = 0$

But $\{a, b, c\}$ are linearly independent. Therefore the only way for the above to happen is if all coefficients of a, b, and c are zero:

$$\begin{cases} \alpha' + \beta' + \gamma' = 0 \\ \beta' + \gamma' = 0 \\ \gamma' = 0 \end{cases} \Rightarrow \begin{cases} \alpha' = 0 \\ \beta' = 0 \\ \gamma' = 0 \end{cases}$$

In summary, we have shown that the only way to make

$$\alpha' u + \beta' v + \gamma' w = 0$$

is by setting $\alpha' = \beta' = \gamma' = 0$. This means that $\{u, v, w\}$ are <u>lin. independent</u>

Problem B. [**60 points**]

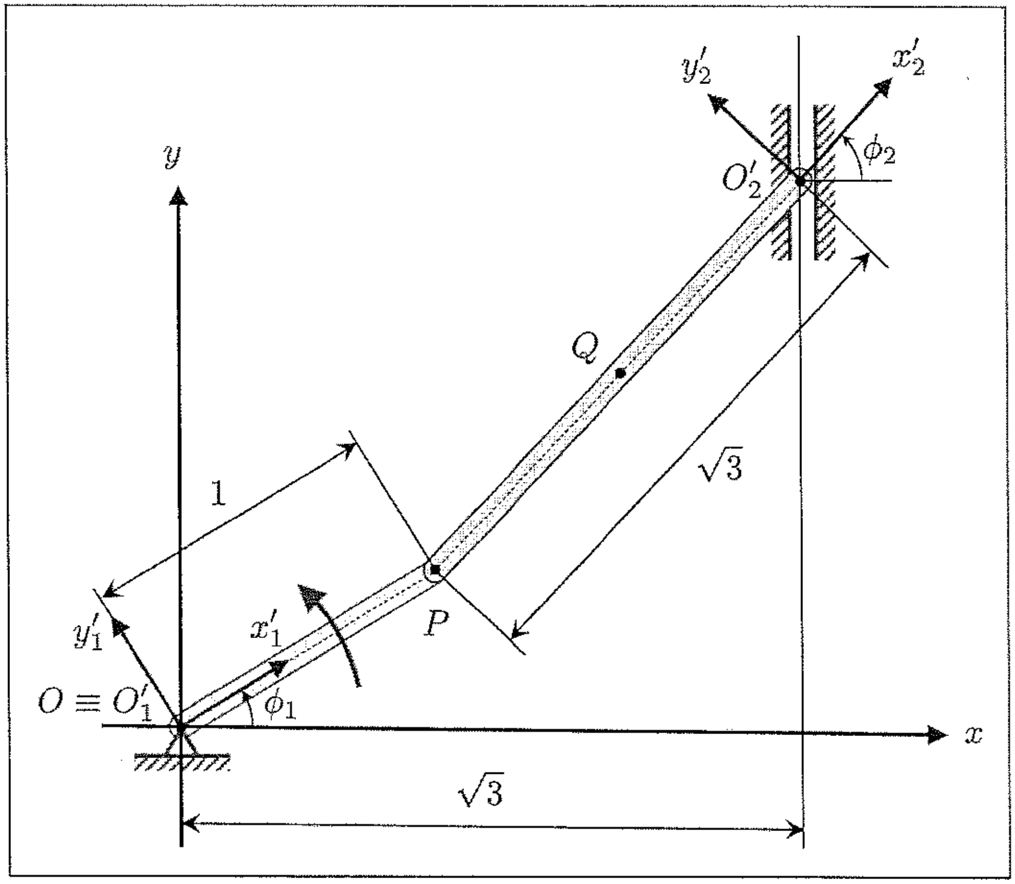

Figure 1: Two body mechanism

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$$\mathbf{q} = [x_1\,, y_1\,, \phi_1\,, x_2\,, y_2\,, \phi_2]\ ,$$

with the local reference frames for the two bodies located as indicated in the figure. Assume further that a motion is prescribed as

$$\phi_1 = \frac{\pi}{6} + \sin(2t)\,.$$

All dimensions, as indicated in the figure are in SI units.

1. (10 points)

   Describe (in *words*) the **kinematic constraints** you would use to model this mechanism. How many (kinematic) degrees of freedom does the mechanism have?

2. (15 points)

   Using the set of generalized coordinates indicated above, specify the set of kinematic constraint equations $\mathbf{\Phi}^K(\mathbf{q}) = \mathbf{0}$ that model this mechanism.

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3. (5 points)

   Specify the driver constraint equations $\Phi^D(\mathbf{q}, t) = \mathbf{0}$, required to impose the desired motion. How many such constraints did you impose and why?

4. (20 points)

   Find the generalized coordinates $\mathbf{q}$ and generalized velocities $\dot{\mathbf{q}}$ at time $t = 0$. In other words, solve the *position analysis* and *velocity analysis* problems at the initial time $t = 0$.

5. (10 points)

   Consider the point Q located at the center of body 2 (assumed to be symmetric). Calculate the position and velocity of point Q at the initial configuration calculated above (at $t = 0$).

6. (Bonus: 15 points)

   At time $t = 0.01s$, carry out the first iteration of the Newton-Raphson iterative process that you would use in performing *position analysis.* As a starting point, use the values of the generalized coordinates at time $t = 0$ (that is, the initial configuration calculated above).

   Notes:

   - You can (but are not required to) use a reduced set of generalized coordinates, $\mathbf{q} = [\phi_1, y_2, \phi_2]$.
   - You do **not** have to solve for the first approximation of the generalized states $\mathbf{q}(t_1)$. Simply write down the equation that you would have to solve in order to calculate them.

7. (Bonus: 15 points)

   Is it possible for this mechanism to reach a singular configuration? Why or why not?

1 The physical constraints we must model are as follow:
- point O_1' on body 1 is constrained to always be at the origin of the GRF. This can be accomplished using an Absolute-X constraint and an Absolute-Y constraint
- point P on body 1 and point P on body 2 are constrained to always coincide. In other words, there is a revolute joint connecting the two points on the two bodies
- point O_2' on body 2 is constrained to only move on the line $x = \sqrt{3}$ (as expressed in the GRF). This can be modeled using an Absolute-X constraint.

The mechanism has a single kinematic degree of freedom:

KDOF = 1

(2) We have

$$\Phi^K(q) = \begin{bmatrix} \Phi^{ax(1)} \\ \Phi^{ay(1)} \\ \Phi^{r(1,2)} \\ \Phi^{ax(2)} \end{bmatrix}$$

where the attributes of the various constraints are as follows:

$\Phi^{ax(1)}$: Absolute-X constraint ~~betwe~~ for point $s_1'^{O} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ using $x_{Ground} = 0$

$\Phi^{ay(1)}$: Absolute-Y constraint for point $s_1'^{O} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ using $y_{Ground} = 0$

$\Phi^{r(1,2)}$: Revolute joint constraint between bodies 1 and 2 using $s_1'^{P} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $s_2'^{P} = \begin{bmatrix} -\sqrt{3} \\ 0 \end{bmatrix}$

$\Phi^{ax(2)}$: Absolute-X constraint for point $s_2'^{O} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ using $x_{Ground} = \sqrt{3}$

As such, the kinematic constraint equations, using the full set of absolute (Cartesian) coordinates, are:

$$\Phi^K(q) = \begin{bmatrix} \left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + A_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) - \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + A_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) - \left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + A_2 \begin{bmatrix} -\sqrt{3} \\ 0 \end{bmatrix} \right) \\ x_2 - \sqrt{3} \end{bmatrix} \begin{array}{l} \leftarrow \Phi^{ax(1)} \\ \leftarrow \Phi^{ay(1)} \\ \leftarrow \Phi^{r(1,2)} \\ \leftarrow \Phi^{ax(2)} \end{array}$$

$$\Phi^K(q) = \begin{bmatrix} x_1 \\ y_1 \\ x_1 + \cos\varphi_1 - x_2 + \sqrt{3}\cos\varphi_2 \\ y_1 + \sin\varphi_1 - y_2 + \sqrt{3}\sin\varphi_2 \\ x_2 - \sqrt{3} \end{bmatrix}$$

Note that we wrote 5 kinematic constraints involving 6 generalized coordinates

$\Rightarrow$ KDOF = 1

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(3) In order to get NDOF = 0, since KDOF = 1, we must specify exactly one driver constraint. In this case, this will be an Absolute Angle driver:

$$\Phi^D(q,t) = \varphi_1 - \frac{\pi}{6} - \sin(2t)$$

(4) Solve $\Phi(q,t) \equiv \begin{bmatrix} \Phi^K(q) \\ \Phi^D(q,t) \end{bmatrix}$ at $t=0$ to find $q_0 = q(0)$.

In other words, we must solve the nonlinear system $\Phi(q_0, 0) = 0$ for q_0

$$\begin{cases} x_1 = 0 \\ y_1 = 0 \\ x_1 + \cos\varphi_1 - x_2 + \sqrt{3}\cos\varphi_2 = 0 \\ y_1 + \sin\varphi_1 - y_2 + \sqrt{3}\sin\varphi_2 = 0 \\ x_2 - \sqrt{3} = 0 \\ \varphi_1 - \frac{\pi}{6} = 0 \end{cases} \Rightarrow$$

$x_1 = 0 \quad y_1 = 0 \quad x_2 = \sqrt{3} \quad \varphi_1 = \frac{\pi}{6}$

From the 3rd eq. we get:

$0 + \frac{\sqrt{3}}{2} - \sqrt{3} + \sqrt{3}\cos\varphi_2 = 0 \Rightarrow \varphi_2 = \frac{\pi}{3}$

From the 4th eq. we get:

$0 + \frac{1}{2} - y_2 + \sqrt{3}\cdot\frac{\sqrt{3}}{2} = 0 \Rightarrow y_2 = 2$

Therefore: $q_0 = \left[0, 0, \frac{\pi}{6}, \sqrt{3}, 2, \frac{\pi}{3}\right]^T$

Taking the time derivative of $\Phi(q,t) = 0$ we obtain $\dot{\Phi} = 0$ which we then evaluate at $t=0$ and solve for $\dot{q}_0 = \dot{q}(0)$:

$$0 = \dot{\Phi} = \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{x}_1 - \dot{\varphi}_1\sin\varphi_1 - \dot{x}_2 - \sqrt{3}\dot{\varphi}_2\sin\varphi_2 \\ \dot{y}_1 + \dot{\varphi}_1\cos\varphi_1 - \dot{y}_2 + \sqrt{3}\dot{\varphi}_2\cos\varphi_2 \\ \dot{x}_2 \\ \dot{\varphi}_1 - 2\cos(2t) \end{bmatrix}$$

When evaluated at $t=0$, we immediately obtain

$\dot{x}_1 = 0 \quad \dot{y}_1 = 0 \quad \dot{x}_2 = 0 \quad \dot{\varphi}_1 = 2$

From the 3rd equation above, we get $0 - 2\cdot\frac{1}{2} - 0 - \sqrt{3}\dot{\varphi}_2\frac{\sqrt{3}}{2} = 0 \Rightarrow \dot{\varphi}_2 = -\frac{2}{3}$

From the 4th equation above, we get $0 + 2\cdot\frac{\sqrt{3}}{2} - \dot{y}_2 - \frac{2}{3}\sqrt{3}\cdot\frac{1}{2} = 0 \Rightarrow \dot{y}_2 = \frac{2\sqrt{3}}{3}$

Therefore: $\dot{q}_0 = \left[0, 0, 2, 0, \frac{2\sqrt{3}}{3}, -\frac{2}{3}\right]^T$

(5) Once the generalized coordinates q and generalized velocities $\dot{q}$ are known at a given time t, the position and velocity of some point Q on body 2, specified through $s_2'^{Q} = \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ 0 \end{bmatrix}$ are obtained using:

$$r_2^{Q} = r_2 + A_2 s_2'^{Q}$$

$$\dot{r}_2^{Q} = \dot{r}_2 + \dot{\varphi}_2 B_2 s_2'^{Q}$$

At time $t=0$ we therefore have:

$$r_2^{Q} = \begin{bmatrix} \sqrt{3} \\ 2 \end{bmatrix} + \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} -\sqrt{3}/2 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{3} - \sqrt{3}/4 \\ 2 - 3/4 \end{bmatrix} = \begin{bmatrix} \frac{3\sqrt{3}}{4} \\ \frac{5}{4} \end{bmatrix}$$

$$\dot{r}_2^{Q} = \begin{bmatrix} 0 \\ \frac{2\sqrt{3}}{3} \end{bmatrix} - \frac{2}{3} \begin{bmatrix} -\sqrt{3}/2 & -1/2 \\ 1/2 & -\sqrt{3}/2 \end{bmatrix} \begin{bmatrix} -\sqrt{3}/2 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \cdot \frac{3}{4} \\ \frac{2\sqrt{3}}{3} + \frac{2}{3}\frac{\sqrt{3}}{4} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{5\sqrt{3}}{6} \end{bmatrix}$$

(6) The system Jacobian is:

$$\Phi_q = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -\sin\varphi_1 & -1 & 0 & -\sqrt{3}\sin\varphi_2 \\ 0 & 1 & \cos\varphi_1 & 0 & -1 & \sqrt{3}\cos\varphi_2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

The first iteration of the Newton-Raphson algorithm solves the equation $\Phi_q(q_0, 0) \cdot \Delta q^{(0)} = -\Phi(q_0, 0)$ for the update $\Delta q^{(0)}$ (where $q^{(0)} = q_0$ as obtained in (4)) then generates the new approximation $q^{(1)} = q^{(0)} + \Delta q^{(0)}$

In our case, $\Delta q^{(0)}$ is the solution of the system:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1/2 & -1 & 0 & -3/2 \\ 0 & 1 & \sqrt{3}/2 & 0 & -1 & \sqrt{3}/2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \Delta q^{(0)} = - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\sin(0.02) \end{bmatrix}$$

(7) With the Jacobian given as in (6) above, its determinant is obtained as

$$\det(\Phi_q) = \sqrt{3} \sin \varphi_2$$

Therefore, the mechanism would reach a singular configuration when the angle φ_2 becomes 0 or π. Note that φ_2 can never be π. Moreover $\varphi_2 = 0$ only if $\varphi_1 = \frac{\pi}{2}$ (as can be observed directly or obtained from the constraint equations). But, the specified driver constraint $\left(\varphi_1 = \frac{\pi}{6} + \sin(2t)\right)$ is such that the angle φ_1 can never reach a value $\frac{\pi}{2}$. Indeed:

$$\varphi_1 = \frac{\pi}{6} + \sin(2t) \leq \frac{\pi}{6} + 1 < \frac{\pi}{2}$$

As such, the mechanism can never reach a singular configuration