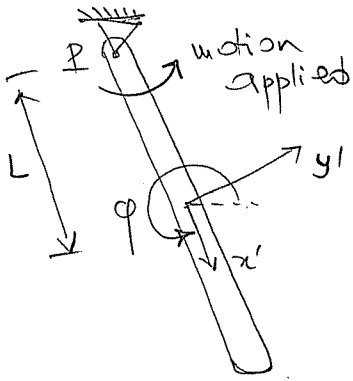


Example
 simple pendulum



A torque is applied to drive the pendulum like $\varphi = \frac{3\pi}{2} + 2\pi t$.

Mass $\rightarrow m$

Mass moment of inertia $\rightarrow J'$

$$M = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J' \end{bmatrix}$$

Constraints (kinematic and driving):

Absolute x & Absolute y constraints.

$$R^p = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c\varphi & -s\varphi \\ s\varphi & c\varphi \end{bmatrix} \begin{bmatrix} -L \\ 0 \end{bmatrix} = \begin{bmatrix} x - Lc\varphi \\ y - Ls\varphi \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then,

$$\Phi(q, t) = \begin{bmatrix} x - Lc\varphi \\ y - Ls\varphi \\ \varphi - 2\pi t - \frac{3\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Phi_q = \begin{bmatrix} 1 & 0 & Ls\varphi \\ 0 & 1 & -Lc\varphi \\ 0 & 0 & 1 \end{bmatrix}$$

Generalized applied force: $Q^A = \begin{bmatrix} 0 \\ -mg \\ 0 \end{bmatrix}$

Then, the EOM assumes the form (based on $M\ddot{q} + \Phi_q^T \lambda = Q^A$)

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J' \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\varphi} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ Ls\varphi & -Lc\varphi & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -mg \\ 0 \end{bmatrix}$$

I need the acceleration constraint equation:

$$\ddot{\Phi}(q, t) = \begin{bmatrix} \ddot{x} + \ddot{\varphi} Ls\varphi \\ \ddot{y} - \ddot{\varphi} Lc\varphi \\ \ddot{\varphi} - 2\pi \end{bmatrix} \quad \ddot{\Phi}(q, t) = \begin{bmatrix} \ddot{x} + \ddot{\varphi} Ls\varphi + \dot{\varphi}^2 Lc\varphi \\ \ddot{y} - \ddot{\varphi} Lc\varphi + \dot{\varphi}^2 Ls\varphi \\ \ddot{\varphi} - 2\pi \end{bmatrix}$$

Therefore, the RHS of the acceleration equation is

$$r = \begin{bmatrix} -\ddot{\varphi}^2 L c \varphi \\ -\ddot{\varphi}^2 L s \varphi \\ 0 \end{bmatrix}$$

We can now assemble the linear system that helps us get the acceleration and λ :

$$\begin{array}{ccc|ccc} \begin{matrix} M \\ \Phi_q^T \end{matrix} & \begin{matrix} u & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & J' \end{matrix} & \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} & \begin{matrix} 0 & 0 & 0 \\ L s \varphi & -L c \varphi & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\varphi}_1 \end{matrix} & = & \begin{matrix} 0 \\ -u g \\ 0 \\ -\ddot{\varphi}^2 L c \varphi \\ -\ddot{\varphi}^2 L s \varphi \\ 0 \end{matrix} \end{array}$$

Fortunately enough, you can solve this linear system without resorting to a numerical approach. You get:

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\varphi}_1 \end{bmatrix} = \begin{bmatrix} -L \cos \varphi \cdot \ddot{\varphi}^2 \\ -L s \varphi \cdot \ddot{\varphi}^2 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} u L \ddot{\varphi}^2 \cdot c \varphi \\ -u g + u L \ddot{\varphi}^2 s \varphi \\ -u g L \cdot c \varphi \end{bmatrix}$$

Next, let's compute the reaction force induced by this pin joint.

To this end, let's identify first Φ_r and Φ_φ :

$$\Phi_r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \Phi_\varphi = \begin{bmatrix} L s \varphi \\ -L c \varphi \\ 1 \end{bmatrix}$$

According to what we discussed in class,

$$F^p = -\Phi_r^T \lambda \quad \& \quad T = -S^{pT} \cdot B^T \cdot F^p - \Phi_\varphi^T \cdot \lambda$$

Then,

$$F^P = - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} mL \dot{\varphi}^2 c\varphi \\ -mg + mL \dot{\varphi}^2 s\varphi \\ -mgL c\varphi \end{bmatrix} = - \begin{bmatrix} mL \dot{\varphi}^2 c\varphi \\ -mg + mL \dot{\varphi}^2 s\varphi \end{bmatrix}$$

For the reaction torque:

$$T = [-L \quad 0] \cdot \begin{bmatrix} -s\varphi & c\varphi \\ -c\varphi & -s\varphi \end{bmatrix} \cdot \begin{bmatrix} mL \dot{\varphi}^2 c\varphi \\ -mg + mL \dot{\varphi}^2 s\varphi \end{bmatrix} - [Ls\varphi \quad -Lc\varphi \quad 1] \begin{bmatrix} mL \dot{\varphi}^2 c\varphi \\ -mg + mL \dot{\varphi}^2 s\varphi \\ -mgL c\varphi \end{bmatrix}$$

$$= [Ls\varphi \quad -Lc\varphi] \begin{bmatrix} mL \dot{\varphi}^2 c\varphi \\ -mg + mL \dot{\varphi}^2 s\varphi \end{bmatrix} - mL^2 \dot{\varphi}^2 s\varphi c\varphi - mgL c\varphi$$

$$+ mL^2 \dot{\varphi}^2 s\varphi c\varphi + mgL c\varphi = mgL c\varphi$$

$$\Rightarrow F^P = - \begin{bmatrix} mL \dot{\varphi}^2 c\varphi \\ -mg + mL \dot{\varphi}^2 s\varphi \end{bmatrix} \quad T = mgL c\varphi. \quad (*)$$



For this simple problem, you know that $\varphi = 2\omega t + \frac{3\pi}{2}$

$$\Rightarrow \dot{\varphi} = 2\omega.$$

Then you can substitute back in Eq. (*) to get:

$$F^P(t) = \begin{bmatrix} -mL (2\omega)^2 \cdot \cos(2\omega t + \frac{3\pi}{2}) \\ mg - mL (2\omega)^2 \sin(2\omega t + \frac{3\pi}{2}) \end{bmatrix} \quad \& \quad T(t) = mgL \cdot \cos(2\omega t + \frac{3\pi}{2})$$

