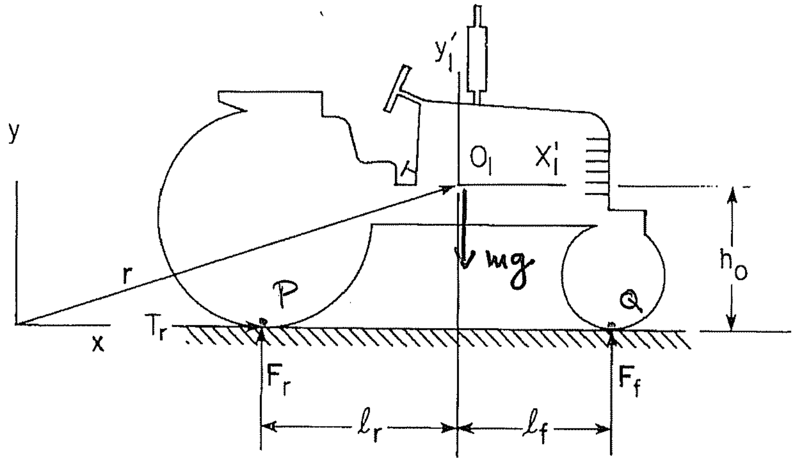
Example 6.1.1



Location of O_1:

$$r = \begin{bmatrix} x \\ y \end{bmatrix}$$

Orientation of local centroidal RF ϕ.

Small angle assumption $\sin\phi \approx \phi$ $\quad \cos\phi \approx 1$

$$r^P = r + A\begin{bmatrix} -l_r \\ -h_0 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 & -\phi \\ \phi & 1 \end{bmatrix}\begin{bmatrix} -l_r \\ -h_0 \end{bmatrix} = \begin{bmatrix} x - l_r + h_0\phi \\ y - l_r\phi - h_0 \end{bmatrix}$$

$$r^Q = r + A\begin{bmatrix} l_f \\ -h_0 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 & -\phi \\ \phi & 1 \end{bmatrix}\begin{bmatrix} l_f \\ -h_0 \end{bmatrix} = \begin{bmatrix} x + l_f + h_0\phi \\ y + l_f\phi - h_0 \end{bmatrix}$$

Deflection in rear tire: $d_r = y^P = y - l_r\phi - h_0$

Deflection in front tire $d_f = y^Q = y + l_f\phi - h_0$.

Assume that both deflections are negative. The force in front and rear tires are:

$$\begin{cases} F_f = -k \cdot d_f = k(h_0 - y - l_f\phi) \\ F_r = -k d_r = k(h_0 - y + l_r\phi) \end{cases}$$

The equations of motion assume the form:

$$M \cdot \ddot{q} = Q \qquad \text{with } Q = Q^P + Q^Q + Q^g.$$

$$\mathcal{Q}^{P} = \begin{bmatrix} F^{P} \\ [B\bar{s}^{P}]^{T} \cdot F^{P} \end{bmatrix}$$

$$s^{P} = \begin{bmatrix} -l_r \\ -h_o \end{bmatrix} \qquad B = \begin{bmatrix} -\varphi & -1 \\ 1 & -\varphi \end{bmatrix} \qquad F^{P} = \begin{bmatrix} T_r \\ F_r \end{bmatrix}$$

$$B\bar{s}^{P} = \begin{bmatrix} -\varphi & -1 \\ 1 & -\varphi \end{bmatrix} \cdot \begin{bmatrix} -l_r \\ -h_o \end{bmatrix} = \begin{bmatrix} h_o + \varphi \cdot l_r \\ h_o\varphi - l_r \end{bmatrix}$$

$$[B\bar{s}^{P}]^{T} \cdot F = T_r(h_o + \varphi \cdot l_r) + F_r(h_o\varphi - l_r)$$

Then $$\mathcal{Q}^{P} = \begin{bmatrix} T_r \\ k(h_o - y + l_r\varphi) \\ T_r(h_o + \varphi l_r) + k(h_o - y + l_r\varphi) \cdot (h_o\varphi - l_r) \end{bmatrix}$$

$$\Longleftrightarrow$$

$$\mathcal{Q}^{\varphi} = \begin{bmatrix} F^{\varphi} \\ [B\bar{s}^{\varphi}]^{T} \cdot F^{\varphi} \end{bmatrix} \quad \text{w/} \quad F^{\varphi} = \begin{bmatrix} 0 \\ k(h_o - y - l_f\varphi) \end{bmatrix} \ \& \ \bar{s}^{\varphi} = \begin{bmatrix} l_f \\ -h_o \end{bmatrix}$$

$$B\bar{s}^{\varphi} = \begin{bmatrix} -\varphi & -1 \\ 1 & -\varphi \end{bmatrix} \begin{bmatrix} l_f \\ -h_o \end{bmatrix} = \begin{bmatrix} h_o - \varphi \cdot l_f \\ h_o\varphi + l_f \end{bmatrix}$$

$$\Rightarrow [B\bar{s}^{\varphi}]^{T} \cdot F^{\varphi} = (h_o\varphi + l_f) \cdot F_f = k(h_o - y - l_f\varphi) \cdot (h_o\varphi + l_f)$$

Then

$$\mathcal{Q}^{\varphi} = \begin{bmatrix} 0 \\ k(h_o - y - l_f\varphi) \\ k(h_o - y - l_f\varphi)(h_o\varphi + l_f) \end{bmatrix}$$

Finally, we have a gravitational force acting at the center of mass and a traction force Tr :

$$Q^g = \begin{bmatrix} 0 \\ -mg \\ 0 \end{bmatrix}$$

Then

$$Q = Q^g + Q^Q + Q^P$$

Therefore, the EOM reads:

$$M \cdot \ddot{q} = Q \quad , \text{ where } \quad M = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J \end{bmatrix} \qquad \ddot{q} = \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\varphi} \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 \\ -mg \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ k(h_0 - y - l_f\varphi) \\ k(h_0 - y - l_f\varphi)(h_0\varphi + l_f) \end{bmatrix} + \begin{bmatrix} Tr \\ k(h_0 - y + l_r\varphi) \\ Tr(h_0 + \varphi l_r) + k(h_0 - y + l_r\varphi)(h_0\varphi - l_r) \end{bmatrix}$$

The textbook linearizes the equations by making the following additional assumptions:

$$h_0\varphi + l_f \approx l_f$$

$$h_0 + \varphi l_r \approx h_0$$

$$h_0\varphi - l_r \approx -l_r$$