Matrix Rank Example

$$J = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 4 & -2 & -2 & 1 \\ 0 & -4 & 0 & 1 \end{bmatrix}$$

(a) What is the row rank of J?

Denote the rows of J by d_1^T, d_2^T, d_3^T ; $d_i \in \mathbb{R}^4$

and notice that

$$-2 \cdot d_1 + 1 \cdot d_2 - d_3 = -2 \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \\ -2 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ -4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, the row rank of J is at most 2.

Consider next a linear combination of the 1^{st} and 3^{rd} rows:

$$a \cdot d_1 + b d_3 = \begin{bmatrix} 2a \\ a \\ -a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -4b \\ 0 \\ b \end{bmatrix}$$

and set it to be the zero vector $[0, 0, 0, 0]^T$.

$$\Rightarrow \begin{cases} 2a = 0 \\ a - 4b = 0 \\ -a = 0 \\ b = 0 \end{cases}$$

which implies $a = b = 0$.

Therefore d_1 and d_3 are linearly independent $\Rightarrow$ row rank of $J = 2$

(b) Column rank = Row rank = 2