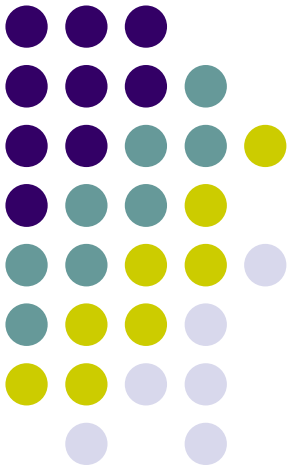
ME451
Kinematics and Dynamics of Machine Systems

Dynamics of Planar Systems

November 22, 2011

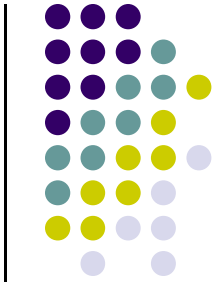
Reaction Forces 6.6

Numerical solution of IVP



"Marge, don't discourage the boy! Weaseling out of things is important to learn.
It's what separates us from the animals! Except the weasel. "
Homer Simpson

Before we get started…

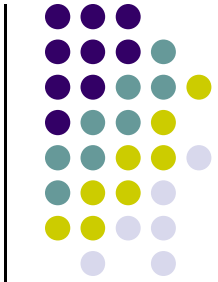


- Last Time
  - Wrap up EOM discussion after we introduce the Lagrange Multiplier Theorem
    - Example covered: EOM for a slider-crank mechanism
  - Discuss why and how to specify Initial Conditions (ICs)
    - Example covered: Pendulum, setting a set of ICs

- Today, we'll learn how to
  - Compute reaction forces that show up in joints
  - Approximate the solution of a differential equation given a set of initial conditions

- Things coming up
  - Second midterm exam is on Th, December 1, at 7:15 PM (room TBA)
    - There will be a review session that day at 5 pm
    - Exam covers the Dynamics component, everything covered *prior* to Dec 1
  - IMPORTANT: we will also have a regular lecture on Dec. 1
    - This is a make-up lecture for Th, December 8
  - There will be no lectures the last week of the semester, when you are supposed to work on your final project and get simEngine2D ready for the final exam (Sat, Dec. 17)

Reaction Forces: The Framework

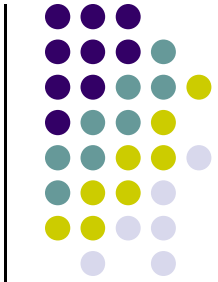


- Remember that we jumped through some hoops to get rid of the reaction forces that would show up in joints

- We'll go back and recover them, since they are important
  - Durability analysis
  - Stress/Strain analysis
  - Selecting bearings in a mechanism
  - Etc.

- It turns out that the key ingredient needed to compute the reaction forces in all joints is the set of Lagrange multipliers λ

Reaction Forces: The Basic Idea



- Recall the partitioning of the total force acting on our mechanical system

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_1 \\ \mathbf{Q}_2 \\ \cdots \\ \mathbf{Q}_{nb} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_1^A & + & \mathbf{Q}_1^C \\ \mathbf{Q}_2^A & + & \mathbf{Q}_2^C \\ \cdots & & \cdots \\ \mathbf{Q}_{nb}^A & + & \mathbf{Q}_{nb}^C \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_1^A \\ \mathbf{Q}_2^A \\ \cdots \\ \mathbf{Q}_{nb}^A \end{bmatrix} + \begin{bmatrix} \mathbf{Q}_1^C \\ \mathbf{Q}_2^C \\ \cdots \\ \mathbf{Q}_{nb}^C \end{bmatrix} = \mathbf{Q}^A + \mathbf{Q}^C$$

- Applying a variational approach (d'Alembert's principle) we ended up with this system-level equation of motion

$$\delta\mathbf{q}^T[\mathbf{M}\ddot{\mathbf{q}} - \mathbf{Q}] = 0 \quad \Rightarrow \quad \mathbf{M}\ddot{\mathbf{q}} - \mathbf{Q} = 0 \quad \Leftrightarrow \quad \mathbf{M}\ddot{\mathbf{q}} - \mathbf{Q}^A - \mathbf{Q}^C = 0$$

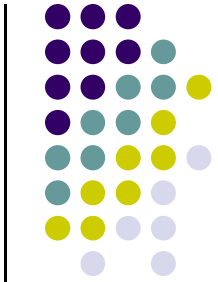
- After jumping through hoops, we ended up with this:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{\Phi}_\mathbf{q}^T\lambda = \mathbf{Q}^A \quad \Leftrightarrow \quad \mathbf{M}\ddot{\mathbf{q}} - \mathbf{Q}^A + \mathbf{\Phi}_\mathbf{q}^T\lambda = \mathbf{0}$$

- Comparing the last two sets of equations it's easy to see that

$$\mathbf{Q}^C = -\mathbf{\Phi}_\mathbf{q}^T\lambda$$

The Important Observation

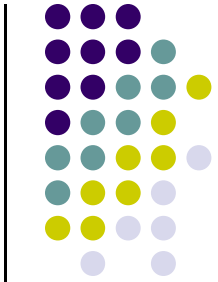


What you get when you computed λ and then premultiply by $\mathbf{\Phi}_{\mathbf{q}}^{T}$ is the constraint reaction force expressed as a **generalized** force:

$$\mathbf{Q}^{C} = -\mathbf{\Phi}_{\mathbf{q}}^{T}\lambda$$

- IMPORTANT OBSERVATION:
  - Actually, you don't care for the "generalized" $\mathbf{Q}^{C}$ flavor of the reaction force, but rather you want a point force & torque combo represented in the Cartesian global reference frame that together produces the $\mathbf{Q}^{C}$ value computed in the equation above
    - You'd like to have F_x, F_y, and a torque T that is due to the constraint
    - You report these quantities as they would act at a point P (it's up to you where you choose this point to be)

The Nuts and Bolts



- The strategy:
  - Look for a point force F (the classical, non-generalized flavor) and a torque T, that when acting on the body would lead to a generalized force equal to $\mathbf{Q}^C$

- Before diving in, remember two things (see November 08 slides)
  - The generalized force associated with a point force was shown to assume the expression

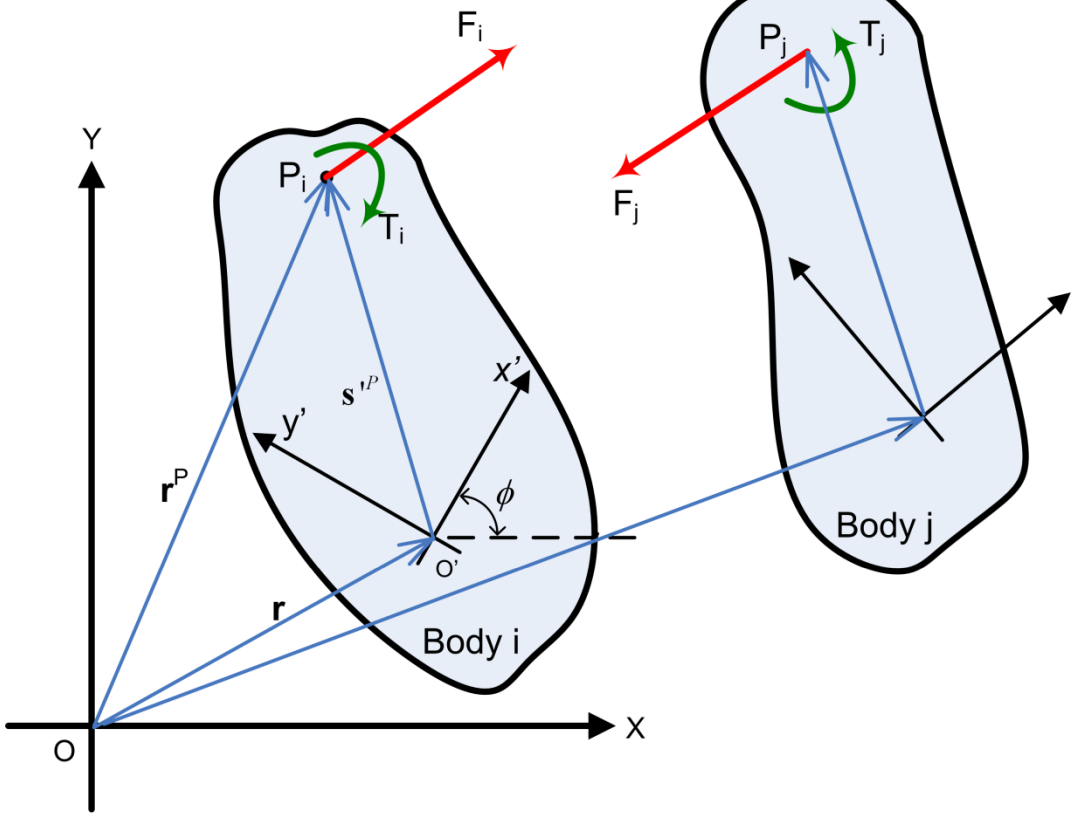
$$\mathbf{Q} = \begin{bmatrix} \mathbf{F}^P \\ (\mathbf{B}\bar{\mathbf{s}}^P)^T \mathbf{F}^P \end{bmatrix}$$

  - The generalized force associated with a concentrated torque acting on a body was shown to assume the expression

$$\mathbf{Q} = \begin{bmatrix} 0 \\ \\ T \end{bmatrix}$$

The Nuts and Bolts

- There is a constraint acting between P_i and P_j and we are after finding the reaction forces/torques $\mathbf{F}_i$ and T_i, as well as $\mathbf{F}_j$ and T_j
- Figure is similar to Figure 6.6.1 out of the textbook



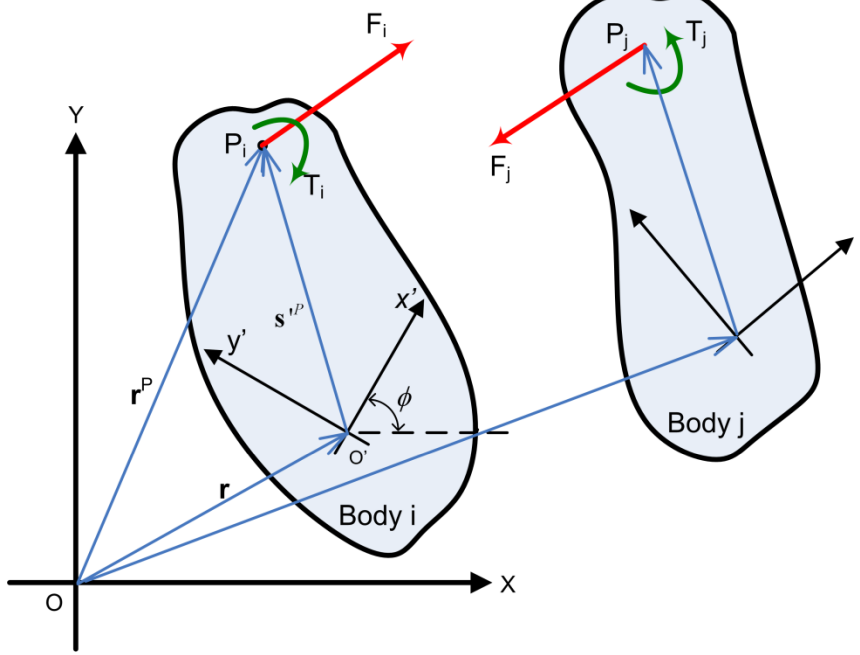
- Textbook covers topic relatively well (pp. 234), I'm only modifying one thing:
  - The book expresses the reaction force/torque $\mathbf{F}_i$ and T_i in a body-fixed reference frame $O'x''y''$ attached at point P_i
  - I didn't see a good reason to do it that way
    - Instead, start by deriving in global reference frame OXY and then multiply by A^T to take it to the body-fixed reference frame

The Main Result

[Expression of reaction force/torque in a constraint]

- Suppose that two bodies *i* and *j* are connected by a joint, and that the equation that describes that joint, which depends on the position and orientation of the two bodies, is

$$\mathbf{\Phi}(\mathbf{q}_i, \mathbf{q}_j, t) = \mathbf{\Phi}(\mathbf{r}_i, \phi_i, \mathbf{r}_j, \phi_j, t) = \mathbf{0}$$



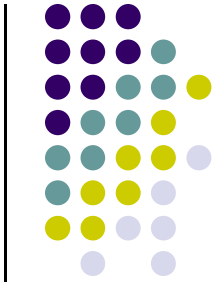
- Suppose that the Lagrange multiplier associated with this joint is $\lambda^{(i\circ j)}$

- Then, the presence of this joint in the mechanism will lead at point P on body *i* to the presence of the following reaction force and torque:

$$\mathbf{F}_i^P = -\mathbf{\Phi}_{\mathbf{r}_i}^T \lambda^{(i\circ j)}$$

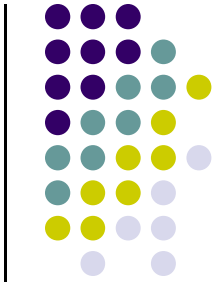
$$T = [(\mathbf{s}'^P_i)^T \mathbf{B}_i^T \mathbf{\Phi}_{\mathbf{r}_i}^T - \mathbf{\Phi}_{\phi_i}^T] \lambda^{(i\circ j)}$$

Comments
(Expression of reaction force/torque in a joint)



- Note that there is a Lagrange multiplier associated with each constraint equation
  - Number of Lagrange multipliers in mechanism is equal to number of constraints
- Each Lagrange multiplier produces (leads to) a reaction force/torque combo
- Therefore, to each constraint equation corresponds a reaction force/torque combo that throughout the time evolution of the mechanism "enforces" the satisfaction of the constraint that it is associated with
  - Example: the revolute joint brings along a set of two kinematic constraints and therefore there will be two Lagrange multipliers associated with this joint
- When the constraint equation acts between two bodies *i* and *j*, there will also be a $\mathbf{F}_j/T_j$ combo associated with each constraint, acting on body j
  - $\mathbf{F}_j$ and T_j are obtained using the equations on previous slide by simply replacing i with j
- Note that you apply the same approach when you are dealing with driving constraints (instead of kinematic constraints)
  - You will get the force and/or torque required to impose that driving constraint

Reaction Forces
~ Remember This ~



- As soon as you have a joint (constraint), you have a Lagrange multiplier $\boldsymbol{\lambda}$

- As soon as you have a Lagrange multiplier you have a reaction force/torque:

$$\mathbf{F}_i^P = -\mathbf{\Phi}_{\mathbf{r}_i}^T \lambda$$

$$T = [(\mathbf{s'}_i^P)^T \mathbf{B}_i^T \mathbf{\Phi}_{\mathbf{r}_i}^T - \mathbf{\Phi}_{\phi_i}^T]\lambda$$

The expression of $\mathbf{\Phi}$ for all the usual joints is known, so a boiler plate approach renders the value of the reaction force in all these joints

- Just in case you want another form for the torque T above, note that

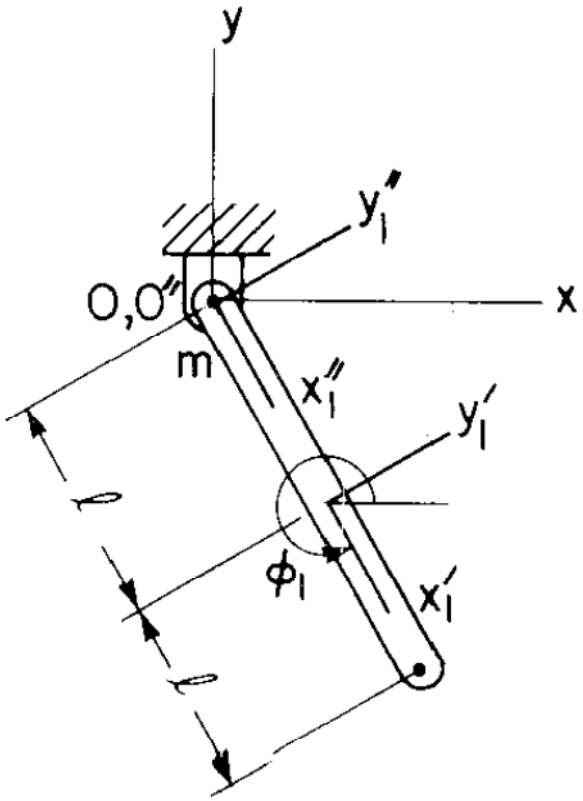
$$T = -(\mathbf{s'}_i^P)^T \mathbf{B}_i^T \mathbf{F}_i^P - \mathbf{\Phi}_{\phi_i}^T \lambda$$

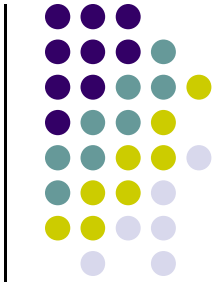
Example 6.6.1: Reaction force in Revolute Joint of a Simple Pendulum

Pendulum driven by motion:

$$\phi_1 = 2\pi t + \frac{3\pi}{2}$$

1) Find the reaction force in the revolute joint that connects pendulum to ground at point O
2) Express the reaction force in the $O'x_1''y_1''$ reference frame

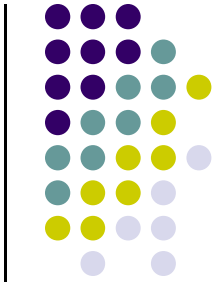




End Constraint Reaction Forces

Numerical Method

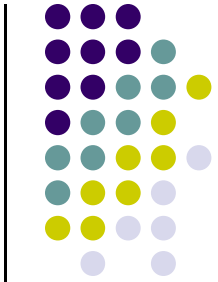
(also called Numerical Algorithm, or simply Algorithm)



- Represents a recipe, a succession of steps that one takes to find an approximation the solution of a problem that otherwise does not admit an analytical solution
  - Analytical solution: sometimes called “closed form” or “exact” solution
  - The approximate solution obtained with the numerical method is also called “numerical solution”

- Examples:
  - Evaluate the integral $$I = \int_0^3 e^{-x^2}\, dx$$
  - Solve the equation $$e^x + \sin(\frac{2}{x}) + \sqrt{x} = 2$$
  - Solve the differential equation that governs time evolution of simple pendulum
  - Many, many others (actually very seldom can you find the exact solution of a problem…)

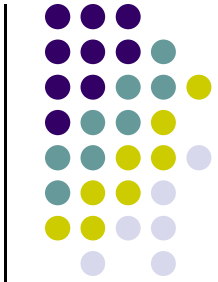
Where/How are Numerical Methods Used?



- Powerful and inexpensive computers have revolutionized the use of numerical methods and their impact
  - Simulation of a car crash in minute detail
  - Formation of galaxies
  - Motion of atoms
  - Finding the electron distribution around nuclei in a nanostructure

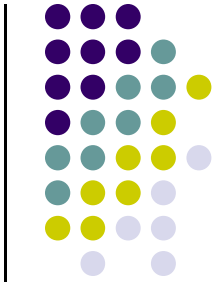
- Numerical methods enable the concept of “simulation-based engineering”
  - You use computer simulation to understand how your mechanism (mechanical system) behaves, how it can be modified and controlled

Numerical Methods in ME451



- In regards to ME451, one would use numerical method to solve the dynamics problem (the resulting set of differential equations that capture Newton's second law)
  - The particular class of numerical methods used to solve differential equations is typically called "**numerical integrators**", or "integration formulas"
  - A numerical integrator generates a numerical solution at **discrete time points** (also called grid points, station points, nodes)
    - This is in fact just like in Kinematics, where the solution is computed on a time grid
  - Different numerical integrators generate **different solutions**, but the solutions are typically very close together, and [hopefully] closed to the actual solution of our problem
  - Putting things in perspective: In 99% of the cases, the use of numerical integrators is **the only alternative** for solving complicated systems described by non-linear differential equations

Numerical Integration
~Basic Concepts~

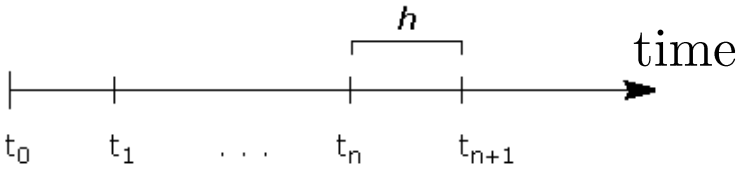


- Initial Value Problem: (IVP)

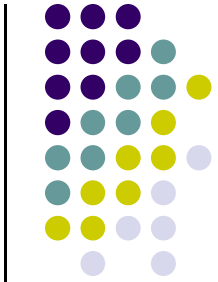
$$\begin{cases} \dot{y} = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

- The general framework
  - You are looking for a function y(t) that depends on time (changes in time), whose time derivative is equal to a function f(t,y) that is given to you (see IVP above)
  - In other words, I give you the derivative of a function, can you tell me what the function is?
  - Remember that both y_0 and the function f are given to you. You want to find y(t).

- In ME451, the best you can hope for is to find an approximation of the unknown function y(t) at a sequence of discrete points (as many of them as you wish)
  - The numerical algorithm produces an approximation of the value of the unknown function y(t) at the each grid point. That is, the numerical algorithm produces $y(t_1)$, $y(t_2)$, $y(t_3)$, etc.



Relation to ME451



- When carrying out Dynamics Analysis, what you can compute is the acceleration of each part in the model
- Acceleration represents the second time derivative of your coordinates
- Somewhat oversimplifying the problem to make the point across, in ME451 you get the second time derivate

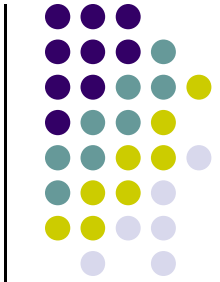
$$\ddot{\mathbf{q}} = \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, t)$$

- This represents a second order differential equation since it has two time derivatives taken on the position **q**

Numerical Integration: Euler's Method

Find solution of this Initial Value Problem:

$$\begin{cases} \dot{y}(t) = f(y,t) \\ y(t_0) = y_0 \end{cases}$$



- The idea: at each grid point t_k, **turn the differential problem into an algebraic problem** by approximating the value of the time derivative:

$$\dot{y}(t_k) \approx \frac{y(t_{k+1}) - y(t_k)}{t_{k+1} - t_k} = \frac{y_{k+1} - y_k}{\Delta t} \quad \Rightarrow \quad y_{k+1} = y_k + \Delta t\, \dot{y}(t_k)$$

This step is called "discretization". It transforms the problem from a continuous **ODE** problem into a discrete **algebraic** problem

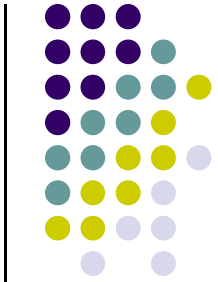
Euler's Method (Δt is the step size):

$$y_{k+1} = y_k + \Delta t\, f(t_k, y_k)$$

Example:

$$\dot{y} = -10\,y$$
$$y(0) = 1$$

- **Integrate 5 steps using Euler's Method**
- **Use an integration step Δt=0.01**
- **Compare to exact solution**



f(t,y) = -10y (no explicit dependency on time t for f in this example)

k=0	y0 = 1.0		
k=1	y1 = y0+f(t0,y0)Δt = 1.0	+ (-10*1.0)*0.01	= 0.9
k=2	y2 = y1+f(t1,y1)Δt = 0.9	+ (-10*.9)*0.01	= 0.81
k=3	y3 = y2+f(t2,y2)Δt = 0.81	+ (-10*.81)*0.01	= 0.729
k=4	y4 = y3+f(t3,y3)Δt = 0.729	+ (-10*.729)*0.01	= 0.6561
k=5	y5 = y4+f(t4,y4)Δt = 0.6561	+ (-10*.6561)*0.01	= 0.5905

Exact solution:
$y(t) = e^{-10t}$

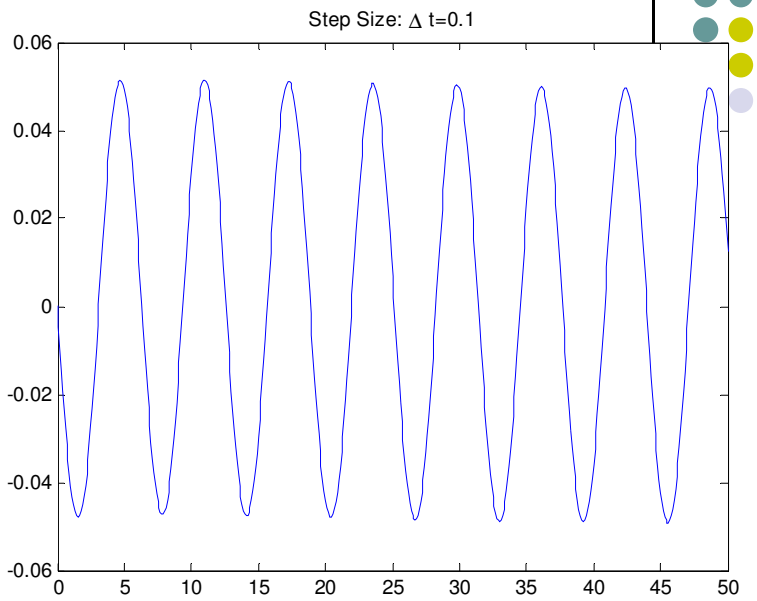
Solution:
y(0) =1.0000
y(0.01)=0.9048
y(0.02)=0.8187
y(0.03)=0.7408
y(0.04)=0.6703
y(0.05)=0.6065

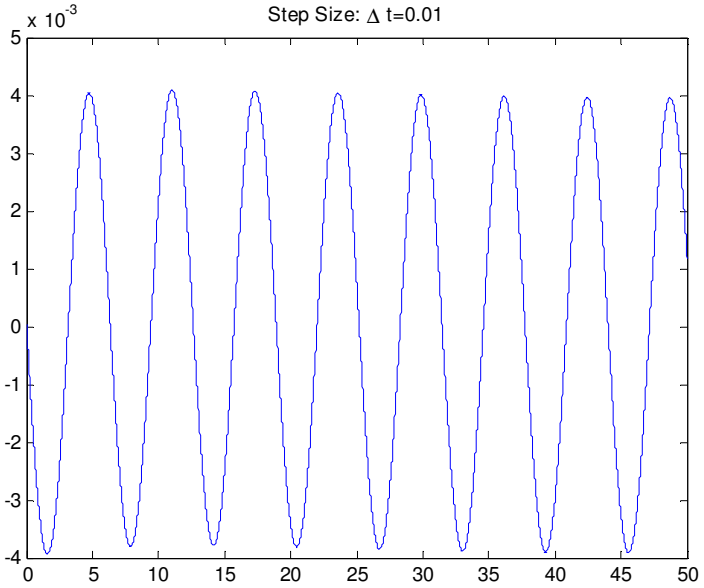
Euler Method:
~ Effect of Step Size ~

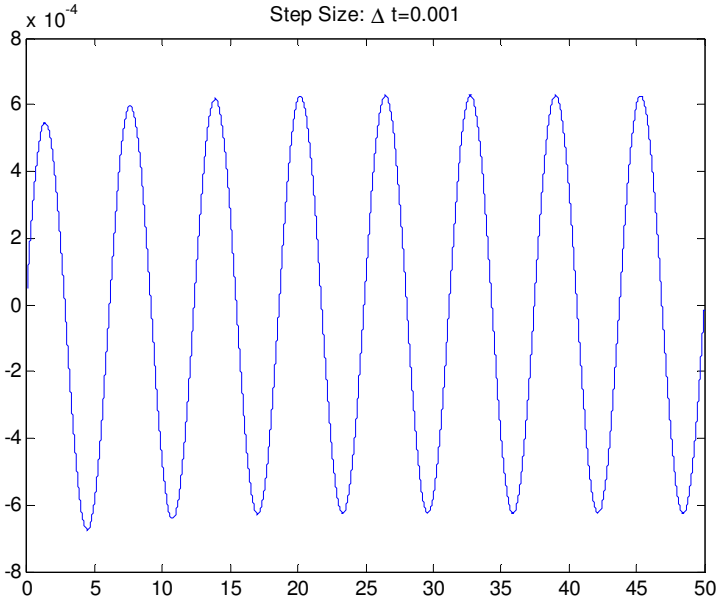
$$\begin{cases} \dot{y} &= -0.1y + \sin t \\ y(0) &= 0 \end{cases}$$

MATLAB code

```
clear
figure
dt = 0.001;
t=0:dt:50;
yExact=.99*exp(-t/10)+.995*sin(t-1.47);
% Numerical solution
th=0:dt:50;
yh=0;
for i=2:length(th)
   f=-yh(i-1)/10+sin(th(i));
   yh(i)=yh(i-1)+f*dt;
end
plot(t,yExact-yh) % this is printing the error…
```



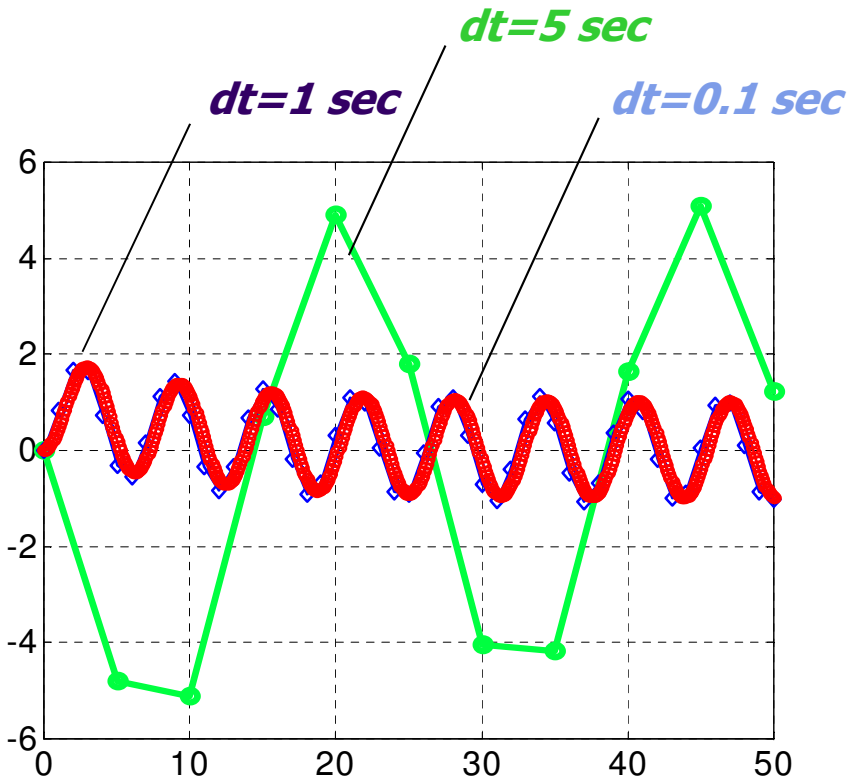




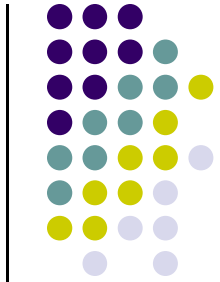
Euler Method:
~ Effect of Step Size ~

$$\begin{cases} \dot{y} &= -0.1y + \sin t \\ y(0) &= 0 \end{cases}$$

- Solve using step sizes *Δt=0.1, 1 and 5 sec*



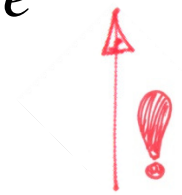
Conclusion: If you use large step-sizes Δt, the **ACCURACY** of the solution is very poor (you can't be too aggressive with size of Δt)

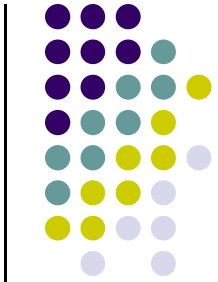


The concept of stiff differential equations, and how to solve the corresponding IVP

Example: IVP

$$\left.\begin{aligned} \dot{y} &= -100\,y \\ y(0) &= 1 \end{aligned}\right\} \Rightarrow \quad y(t) = e^{-100t}$$



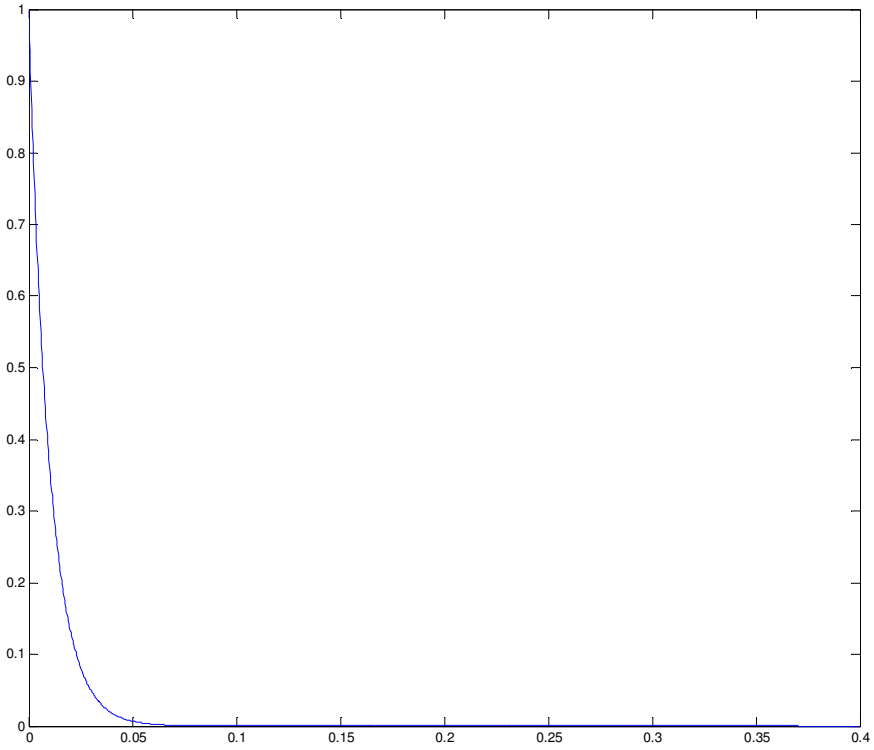


- Integrate 5 steps using forward Euler formula: $\Delta t=0.002$, $\Delta t=0.01$, $\Delta t=0.03$
- Compare the errors between numerical and analytical solutions (Algorithm Error)

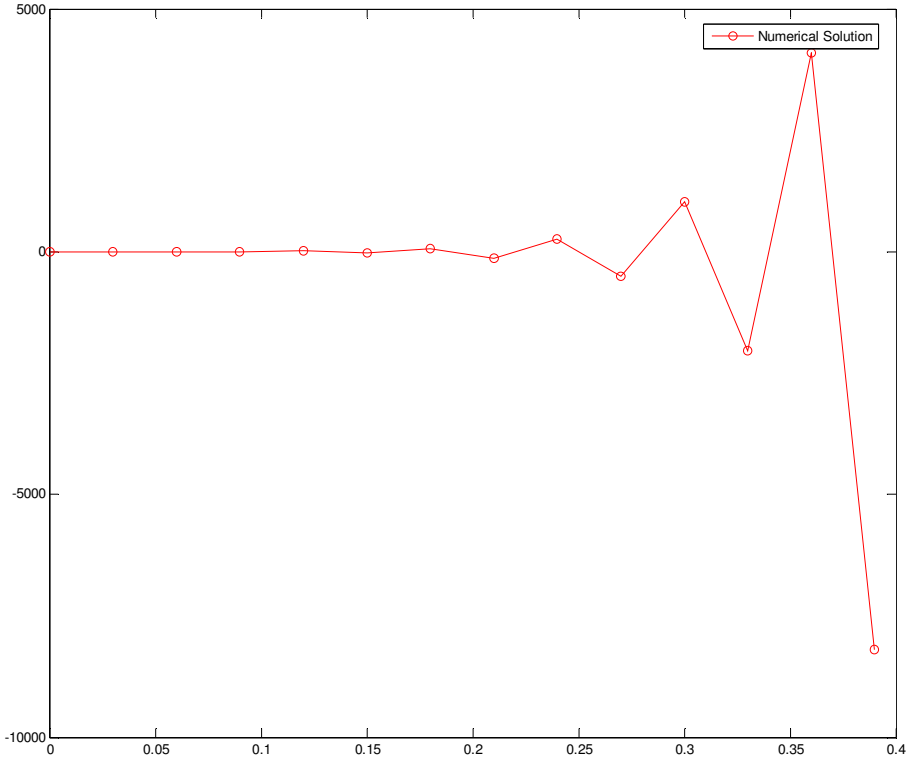
Algorithm Error when $\Delta t=0.002$:	Algorithm Error when $\Delta t=0.01$:	Algorithm Error when $\Delta t=0.03$:
0	0	0
0.01873075307798	0.36787944117144	2.04978706836786
0.03032004603564	0.13533528323661	-3.99752124782333
0.03681163609403	0.04978706836786	8.00012340980409
0.03972896411722	0.01831563888873	-15.99999385578765
0.04019944117144	0.00673794699909	32.00000030590232

Example:

$$\left.\begin{aligned}\dot{y} &= -100\,y \\ y(0) &= 1\end{aligned}\right\} \Rightarrow \quad y(t) = e^{-100t}$$



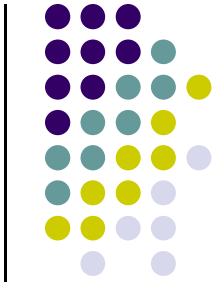
Analytical Solution



Forward Euler

(Δt=0.03)

Concept of stiff IVP's

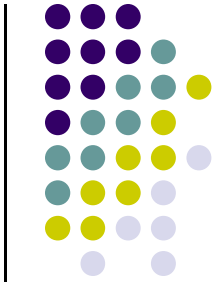


- IVP's for which forward Euler doesn't work well (see example)
  - In general, the entire class of so called **explicit** formulas doesn't work
    - Forward Euler, Runge-Kutta (RK23, RK45), DOPRI5, Adams-Bashforth, etc.

- Stiff IVP's require a different class of integration formulas
  - **Implicit** formulas
    - Example: Backward Euler

$$y_{k+1} = y_k + f(t_{k+1}, y_{k+1})\Delta t$$

Explicit vs. Implicit Formulas (look at Euler family)



- Initial Value Problem

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

- Forward Euler

$$y_{k+1} = y_k + \Delta t\, y'_k$$
$$y'_k = f(y_k, t_k)$$

- Backward Euler

$$y_{k+1} = y_k + \Delta t\, y'_{k+1}$$
$$y'_{k+1} = f(y_{k+1}, t_{k+1})$$

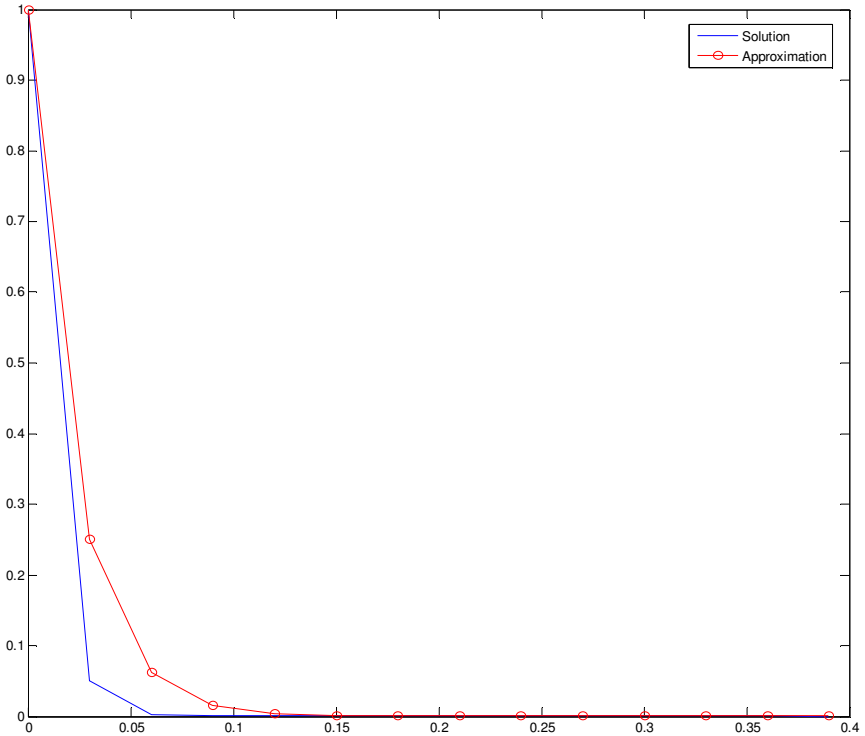
$$\boxed{y_{k+1} = y_k + f(t_k, y_k)\Delta t}$$

$$\boxed{y_{k+1} = y_k + f(t_{k+1}, y_{k+1})\Delta t}$$

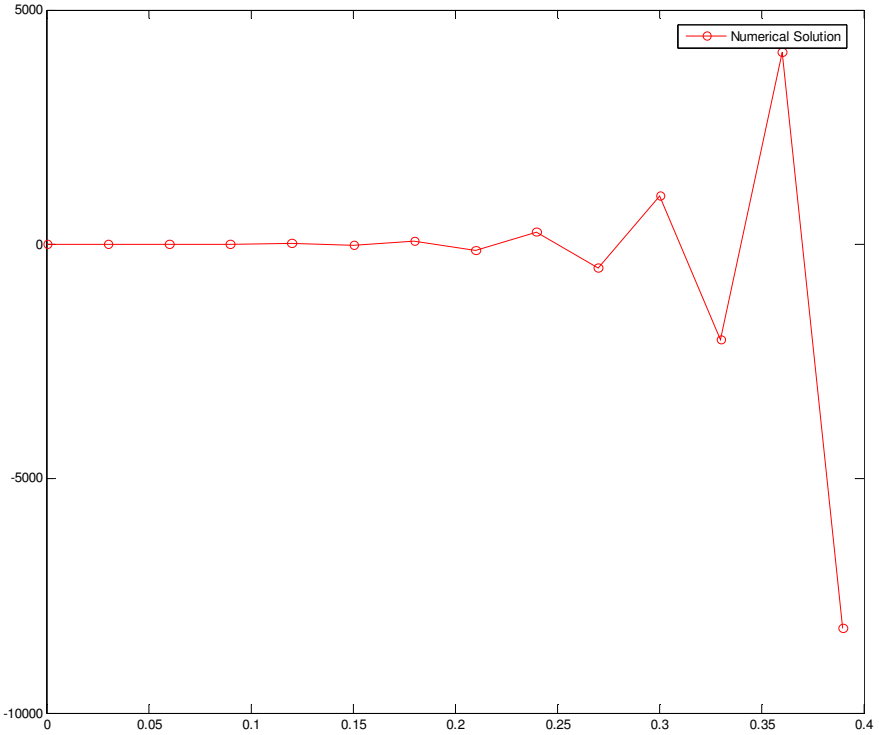
Example:

$$\left.\begin{aligned}\dot{y} &= -100\,y \\ y(0) &= 1\end{aligned}\right\} \Rightarrow \quad y(t) = e^{-100t}$$

Exact Solution



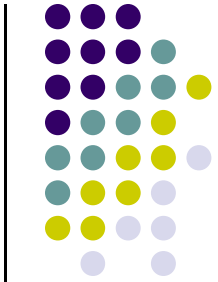
Backward Euler and Analytical Solution



Forward Euler

(Δt=0.03)

Other Popular Algorithms for Stiff IVPs

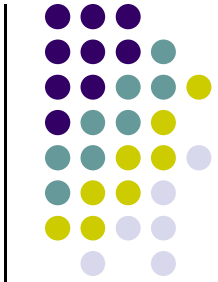


The family of BDF methods (Backward-Difference Formulas):

- BDF of 1st order: $y_{n+1} = y_n + hy'_{n+1}$
- BDF of 2nd order: $y_{n+1} = \frac{4}{3}y_n - \frac{1}{3}y_{n-1} + \frac{2}{3}hy'_{n+1}$
- BDF of 3rd order: $y_{n+1} = \frac{18}{11}y_n - \frac{9}{11}y_{n-1} + \frac{2}{11}y_{n-2} + \frac{6}{11}hy'_{n+1}$
- BDF of 4th order: $y_{n+1} = \frac{48}{25}y_n - \frac{36}{25}y_{n-1} + \frac{16}{25}y_{n-2} - \frac{3}{25}y_{n-3} + \frac{12}{25}hy'_{n+1}$
- BDF of 5th order: $y_{n+1} = \frac{300}{137}y_n - \frac{300}{137}y_{n-1} + \frac{200}{137}y_{n-2} - \frac{75}{137}y_{n-3} + \frac{12}{137}y_{n-4} + \frac{60}{137}hy'_{n+1}$

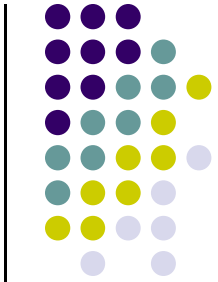
Bill Gear
1970 Inventor of BDF

The Two Key Attributes of a Numerical Integrator



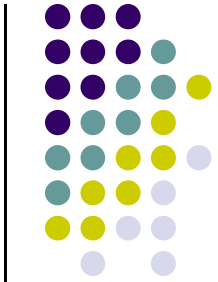
- Two attributes are relevant when considering a numerical integrator for finding an approximation of the solution of an IVP
  - The STABILITY of the numerical integrator
  - The ACCURACY of the numerical integrator

Numerical Integration Formula: The STABILITY Attribute

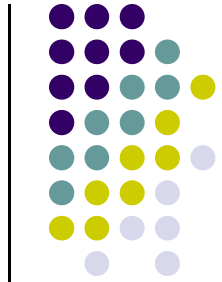


- The stability question:
  - How big can I choose the integration step-size Δt and be safe?
    - Tough question, answered in a Numerical Analysis class
- Different integration formulas, have different stability regions
- You'd like to use an integration formula with large stability region:
  - Example: Backward Euler, BDF methods, Newmark, etc.
- Why not always use these methods with large stability region?
  - There is **no free lunch**: these methods are implicit methods that require the solution of an algebra problem at each step (we'll see this on Th)

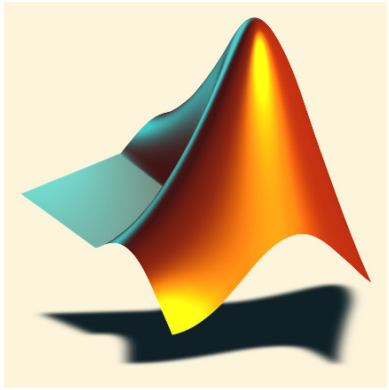
Numerical Integration Formula : The ACCURACY Attribute



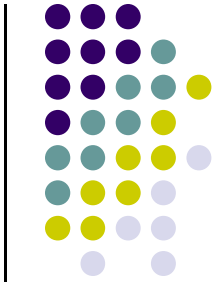
- The accuracy question:
  - How accurate is the formula that I'm using?
  - If I start decreasing Δt , how will the accuracy of the numerical solution improve?
    - Tough question answered in a Numerical Analysis class
- Examples:
  - Forward and Backward Euler: accuracy $O(\Delta t)$
  - RK45: accuracy $O(\Delta t^4)$
- Why not always use methods with high accuracy order?
  - There is **no free lunch**: these methods usually have very small stability regions
  - Therefore, you are limited to very small values of Δt



MATLAB Support for solving IVP
[supplemental material]



Ordinary Differential Equations (Initial Value Problem)



- An ODE + initial value: $\begin{cases} \dot{y} = f(t, y) \\ y(t_0) = y_0 \end{cases}$

- Use ***ode45*** for non-stiff IVPs and ***ode23t*** for stiff IVPs (concept of "stiffness" discussed shortly)

```
[t,y] = ode45(odefun,tspan,y0,options)
```

```
function dydt = odefun(t,y)
```

```
Initial value
```

```
[initialtime  finaltime]
```

  - Use ***odeset*** to define **`options`** parameter above

IVP Example (MATLAB at work):

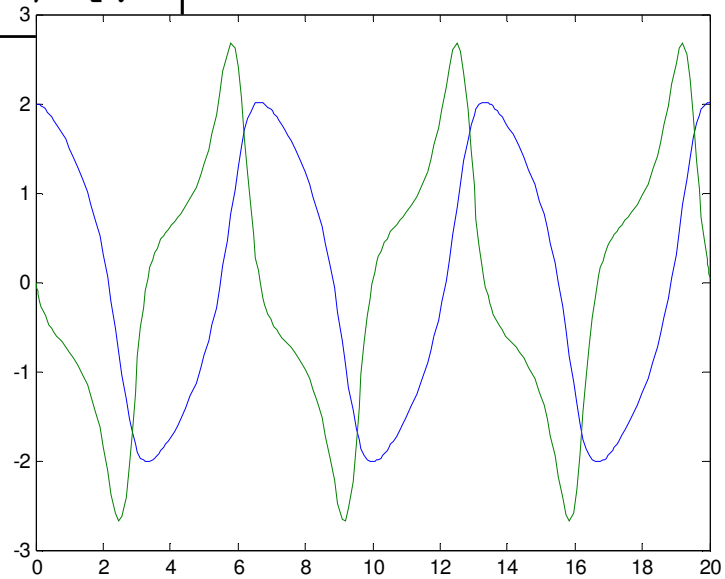
$$\ddot{y}_1 - (1 - y_1^2)\dot{y}_1 + y_1 = 0$$

$$\dot{y}_1 = y_2$$

$$\dot{y}_2 = (1 - y_1^2)y_2 - y_1$$

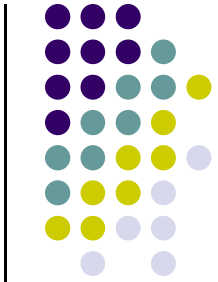
```
function dydt = myfunc(t,y)
dydt=zeros(2,1);
dydt(1)=y(2);
dydt(2)=(1-y(1)^2)*y(2)-y(1);
```

```
» [t,y]=ode45('myfunc',[0 20],[2;0])
```



Note:
Help on ***odeset*** to set options for more **accuracy** and other useful utilities like drawing results during solving.

ODE solvers in MATLAB



Solver	Problem Type	Order of Accuracy	When to use
ode45	Nonstiff	Medium	Most of the time. This should be the first solver tried
ode23	Nonstiff	Low	For problems with crude error tolerances or for solving moderately stiff problems.
ode113	Nonstiff	Low to high	For problems with stringent error tolerances or for solving computationally intensive problems
ode15s	Stiff	Low to medium	If ods45 is slow because the problem is stiff
ode23s	Stiff	Low	If using crude error tolerances to solve stiff systems and the mass matrix is constant
ode23t	Moderately stiff	Low	For moderately stiff problems is you need a solution without numerical damping
ode23tb	Stiff	Low	If using crude error tolerances to solve stiff systems