ME451
Kinematics and Dynamics of Machine Systems

Dynamics of Planar Systems

November 08, 2011

6.1.3, 6.1.4, 6.2, starting 6.3

"If you're going through hell, keep going."
Winston Churchill

Before we get started…

- Last Time
  - Started the derivation of the EOM for one planar rigid body

- Today
  - Finish the derivation of the EOM
  - Understand how the presence of a concentrated force factors into the EOM
  - Look into inertia properties of 2D geometries
    - Center of mass
    - Parallel axis theorem
    - Mass moment of inertia for composite geometries

- Miscellaneous
  - Take-home component of the exam: you have one more week to complete it
    - Now due on Nov. 17 $\rightarrow$ should be a general purpose simEngine2D
    - TA will hold lecture of Tu, Nov. 15: explain how to bridge the parsing and the equation formulation
  - Please see me after class or tomorrow (Wd) during office hours if you think that your exam score doesn't reflect your work

Deriving the EOM for One Rigid Body

- I want to derive this:

$$
\begin{aligned}
m\ddot{\mathbf{r}} - \mathbf{F} &= \mathbf{0} \qquad \text{Equations of Motion governing translation} \\
\bar{J}\ddot{\phi} - n &= 0 \qquad \text{Equation of Motion governing rotation}
\end{aligned}
$$

Notation & Nomenclature

- Matrix-vector notation for the expression of d'Alembert Principle:

$$\delta \mathbf{r}^T[\mathbf{F} - m\ddot{\mathbf{r}}] + \delta\phi[n - J'\ddot{\phi}] = 0$$

$$\Downarrow$$

$$\delta \mathbf{q}^T[\mathbf{Q} - \mathbf{M}\ddot{\mathbf{q}}] = 0$$

$$\mathbf{q} = \begin{bmatrix} \mathbf{r} \\ \phi \end{bmatrix} \qquad \delta\mathbf{q} = \begin{bmatrix} \delta\mathbf{r} \\ \delta\phi \end{bmatrix} \qquad \mathbf{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J' \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} \mathbf{F} \\ n \end{bmatrix}$$

So what do I actually mean when I talk about "generalized forces"?

~ I mean the **Q** above ~

Focusing on F and n…

- **F** was the sum of all distributed forces **f**(P) acting per unit mass:

$$\mathbf{F} = \int_m \mathbf{f}(P)\, \mathrm{d}m(P) \qquad \text{(Eq. 6.1.16)}$$

- **n** was the torque produced by the forces **f**(P)

$$n = \int_m \left(\mathbf{B}\bar{\mathbf{s}}^P\right)^T \mathbf{f}(P)\, \mathrm{d}m(P) = \int_m \left(\bar{\mathbf{s}}^{P^\perp}\right)^T \bar{\mathbf{f}}(P)\, \mathrm{d}m(P) \qquad \text{(Eq. 6.1.17)}$$

- **QUESTION**: What happens when we don't only have distributed forces, such as **f**(P), at each point P on the body, but also a concentrated force acting at only *one* point P of the body?

Handling Concentrated Forces

- The fundamental idea:
  - Whenever some new force shows up, figure out the virtual work that it brings into the picture
  - Then account for this "injection" of virtual work in the virtual work balance equation:

$$\delta W = \delta \mathbf{r}^T [\mathbf{F} - m\ddot{\mathbf{r}}] + \delta\phi[n - J'\ddot{\phi}] = 0$$

  - Caveat: Notice that for **rigid** bodies, the virtual displacements are $\delta\mathbf{r}$ and $\delta\phi$.
    - Some massaging of the additional virtual work might be needed to bring it into the standard form, that is

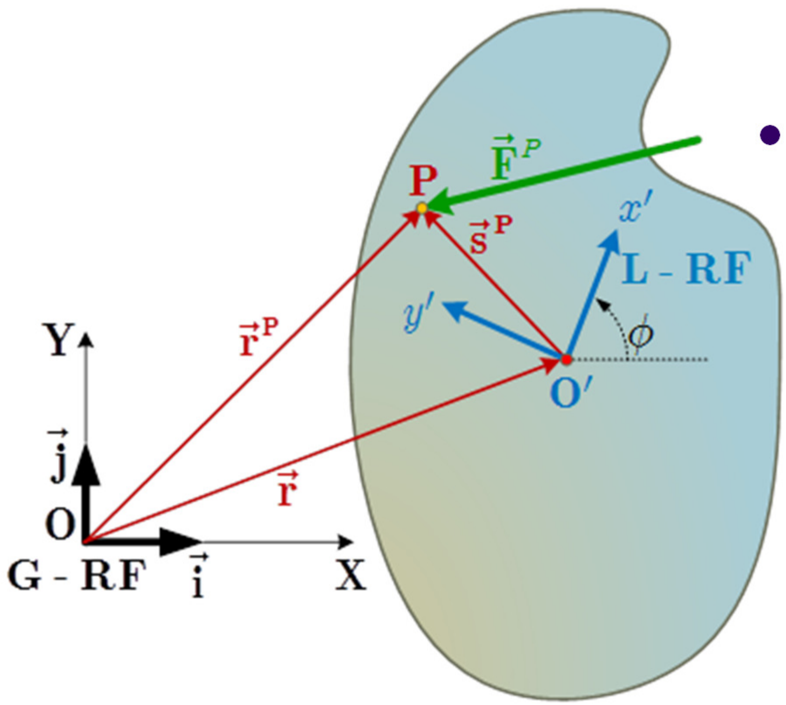
$$\text{Additional } \delta W = \delta \mathbf{r}^T \cdot \underbrace{\text{something1}}_{\text{... goes into } \mathbf{F}} + \delta\phi \cdot \underbrace{\text{something2}}_{\text{... goes into } n}$$

(Keep this in mind when you solve Problem 6.2.1)

[Review of material from last lecture]

Concentrated (Point) Force

- Setup: At a particular point P, you have a point-force $\mathbf{F}^P$ acting on the body



- Two step approach to handle concentrated force:
  - Step A: write the virtual work produced by this force as a results of a virtual displacement of the body
  - Step B: express the force F produced *additional virtual work* in terms of <u>body</u> virtual displacements

$$\delta W = [\delta \mathbf{r}^P]^T \cdot \mathbf{F}^P$$

- Recall:

$$\text{Additional } \delta W = \delta \mathbf{r}^T \cdot \underbrace{\text{something1}}_{\text{... goes into } \mathbf{F}} + \delta\phi \cdot \underbrace{\text{something2}}_{\text{... goes into } n}$$

Concentrated (Point) Force

[Review, cntd.]

- How is virtual work computed?

$$\delta W = [\delta \mathbf{r}^P]^T \cdot \mathbf{F}^P$$

- How is the virtual displacement of point P computed? (we already know this…)

$$\mathbf{r}^P = \mathbf{r} + \mathbf{A}\mathbf{s}'^P$$

$$\Downarrow$$

$$\delta \mathbf{r}^P = \delta \mathbf{r} + \delta\phi \, \mathbf{B}\mathbf{s}'^P$$

- The critical step: expressing the displacement that the force goes through in terms of the body virtual displacements $\delta\mathbf{r}$ and $\delta\phi$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{F}^P \\ (\mathbf{B}\bar{\mathbf{s}})^T \mathbf{F}^P \end{bmatrix}$$

On the meaning of $[\mathbf{B}\bar{\mathbf{s}}^P]^T \cdot \mathbf{F}^P$

- Claim : $[\mathbf{B}\bar{\mathbf{s}}^P]^T \cdot \mathbf{F}^P$ represents the torque of $\mathbf{F}^P$ about the origin of the L-RF. In what follows we'll confirm that this is true.

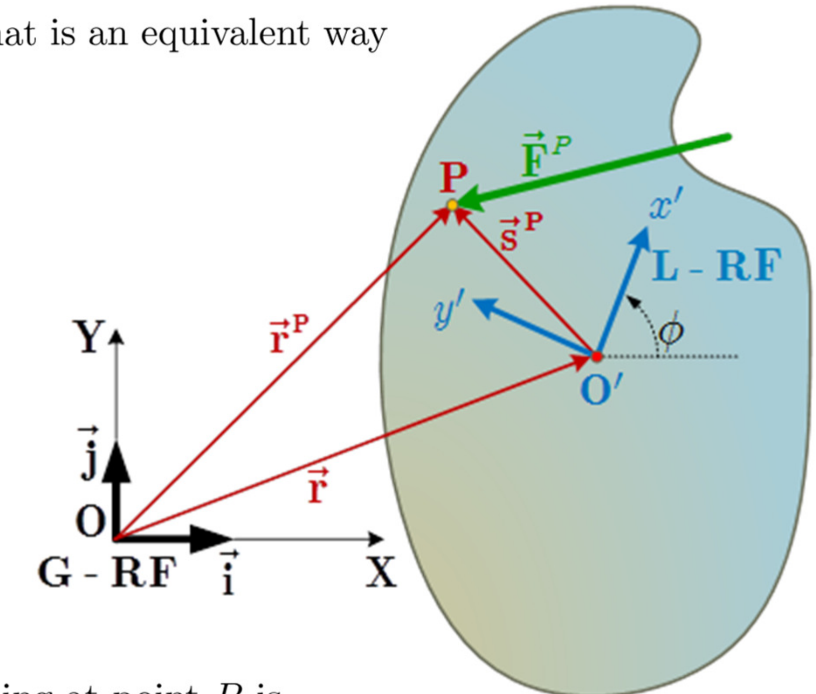
- The size of the torque $\mathbf{r}^P \times \mathbf{F}^P$ is $||\mathbf{s}^P|| \cdot ||\mathbf{F}^P|| \cdot \sin\theta$. What is an equivalent way to express this quantity?

- Use the identity

$$\sin\theta = \cos(\theta - \frac{\pi}{2})$$

- Then, the value of the torque, is

$$\begin{aligned} n^P &= ||\mathbf{s}^P|| \cdot ||\mathbf{F}^P|| \sin\theta \\ &= ||\mathbf{s}^P|| \cdot ||\mathbf{F}^P|| \cos(\theta - \tfrac{\pi}{2}) \\ &= [\mathbf{s}^{P\perp}]^T \cdot \mathbf{F}^P \\ &= [\mathbf{B}\bar{\mathbf{s}}^P]^T \cdot \mathbf{F}^P \end{aligned}$$



- To conclude, the value of the torque produced by $\mathbf{F}^P$ acting at point P is

$$n^P = [\mathbf{B}\bar{\mathbf{s}}^P]^T \cdot \mathbf{F}^P$$

- CONCLUSION: A concentrated (point) force also leads to the presence of a torque that affects the rotation of the body (the rotational degree of freedom associated with the motion of the body).

End: 6.1.3
Begin: 6.1.4

Inertia Properties of Rigid Bodies

- Issues discussed:
  - Determining location of the center of mass (centroid) of a rigid body
  - Parallel axis theorem
  - Mass moment of inertia of composite bodies
  - Location of centroid for a rigid body with a symmetry axis

Location of the Center of Mass

- By <u>definition</u> the centroid is that point of a body for which, when you place a reference frame at that point and compute a certain integral in relation to that reference frame, the integral vanishes:

$$\int_m \mathbf{s}'^P \, dm(P) = 0$$

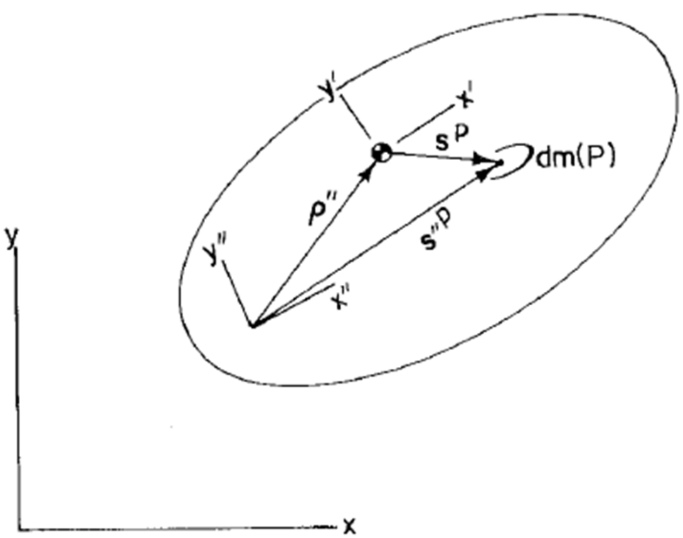


Figure 6.1.3 Location of a centroid.

Q: How can I determine the location of the center of mass?

Mass Moment of Inertia (MMI)

- The MMI was defined as:

$$J' = \int_m \left[\mathbf{s}'^P\right]^T \mathbf{s}'^P \, dm(P)$$

- Suppose that the reference frame with respect to which the integral is computed is a centroidal RF

- You might ask yourself, what happens if I decide to evaluate the integral above with respect to a different reference frame, located at a different point attached to this body?

- Answer is provided by the parallel axis theorem (ρ is vector from centroidal reference frame to the new reference frame) :

$$\boxed{J'_{new} = J' + m\,(\rho^T \rho)}$$

MMI of a Composite Body

- Step 1: Compute the centroid of the composite body

$$\rho'' = \frac{1}{m} \sum_{i=1}^{k} m_i \rho_i''$$

- Step 2: For each sub-body, apply the parallel axis theorem to compute the MMI of that sub-body with respect to the newly computed centroid

$$J^* = \sum_{i=1}^{k} (J_i' + m_i \rho_i^{*T} \rho_i^*)$$

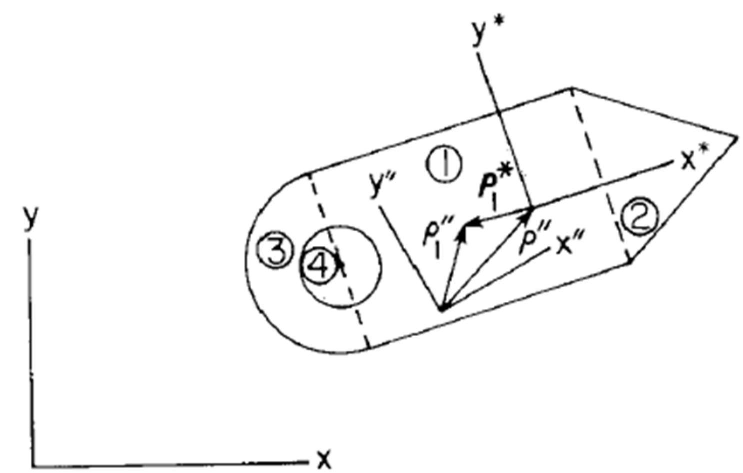


Figure 6.1.5 Body made up of subcomponents.

- Note: if holes are present in the composite body, it's ok to add and subtract material (this translates into positive and negative mass)

Location of the Center of Mass (Cntd.)

- What can one say if the rigid body has a symmetry axis?
  - Here symmetry axis means that **both** mass and geometry are symmetric with respect to that axis

- You can say this: the centroid is somewhere along that axis

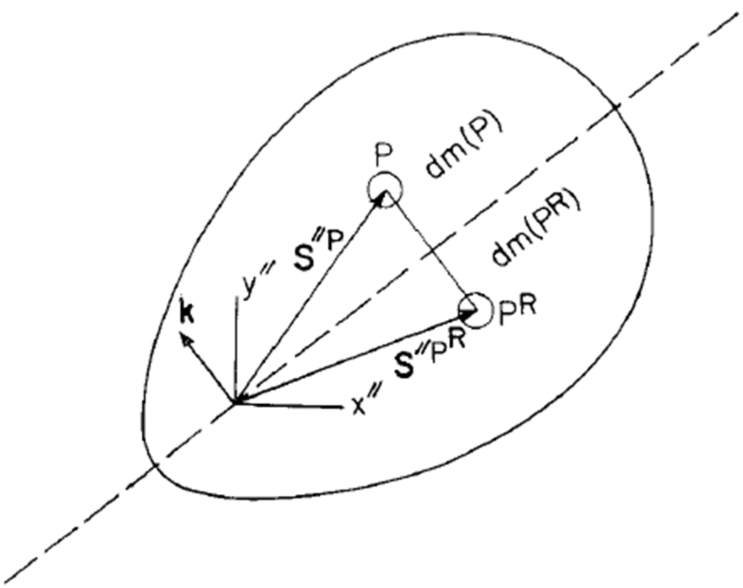


Figure 6.1.4 Body with axis of symmetry.

- NOTE: if the rigid body has two axes of symmetry, the centroid is on each of them, and therefore is where they intersect

What's Left ?
30,000 Feet Perspective

- Two important issues remain to be addressed:

1) Elaborate on the nature of the "concentrated forces" that we have introduced. A closer look at the nature of these "concentrated" forces reveals that they could be
   - Forces coming out of translational spring-damper-actuator elements
   - Forces coming out of rotational spring-damper-actuator elements
   - Reaction forces (due to the presence of a constraint, say between body and ground)

2) We only derived the variational form of the equation of motion for the trivial case of *one* rigid body. How do I derive the variational form of the equations of motion for a mechanism with many components (bodies) connected through joints?
   - Just like before, we'll rely on the principle of virtual work

- Where are we going with this? Why do we need 1) and 2)?
  - We have to formulate the equations that govern the time evolution of an arbitrary collection of rigid bodies interconnected by an arbitrary set of kinematic constraints.