ME451
Kinematics and Dynamics of Machine Systems

Newton-Raphson Method 4.5

Closing remarks, Kinematics Analysis

October 25, 2010

"Success is going from failure to failure without loss of enthusiasm."
Winston Churchill

Before we get started…

- Last Time
  - Discussion about visualizing the time evolution (motion) of your mechanism
  - Discussed wrecker-boom example

- Today:
  - Discuss Newton-Raphson Method (See section 4.5 – needed for take-home)
  - Closing the discussion on Kinematics Analysis
    - Discuss Position Analysis, Velocity Analysis, and Acceleration Analysis

- Next week:
  - Midterm on Th
  - Review on Wd evening, 6 PM

Solving a Nonlinear System

- The most important numerical algorithm to understand in Kinematics

- Relied upon heavily by ADAMS, used almost in <u>all</u> analysis modes
  - Kinematics
  - Dynamics
  - Equilibrium

- **Q**: How does one go about finding the solution of problems such as these?

$$\sqrt{x} - \sin x = 0$$

$$\begin{cases} 3x + \sin(xy) - 4 & = & 0 \\ x - e^{\cos(y)} & = & 0 \end{cases}$$

$$\begin{cases} x - e^{y} - 1 & = & 0 \\ \ln(1+x) - \cos y & = & 0 \end{cases}$$

Newton-Raphson Method

- Start by looking at the one-dimension case:
  - Find the solution x^* of the equation:

$$f(x) = 0$$

  - Assumption:
    - The function $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable
    - Example on previous slide:

$$f(x) = \sqrt{x} - \sin x$$

Newton-Raphson Method

- Algorithm
  - Start with <u>initial guess</u> x$^{(0)}$ and then compute x$^{(1)}$, x$^{(2)}$, x$^{(3)}$, etc.

$$x^{(1)} = x^{(0)} - \frac{f(x^{(0)})}{f'(x^{(0)})} = x^{(0)} - [f'(x^{(0)})]^{-1} \cdot f(x^{(0)})$$

$$x^{(2)} = x^{(1)} - \frac{f(x^{(1)})}{f'(x^{(1)})} = x^{(1)} - [f'(x^{(1)})]^{-1} \cdot f(x^{(1)})$$

$$x^{(3)} = x^{(2)} - \frac{f(x^{(2)})}{f'(x^{(2)})} = x^{(2)} - [f'(x^{(2)})]^{-1} \cdot f(x^{(2)})$$

$$\ldots$$

- This is called an iterative algorithm:
  - First you get x$^{(1)}$, then you improve the predicted solution to x$^{(2)}$, then improve more yet to x$^{(3)}$, etc.
  - Iteratively, you keep getting closer and closer to the actual solution

Example: Solve f(x)=0

$$f(x) = x^2 + \sin x - 1.841471 = 0$$

$$f'(x) = 2x + \cos x$$

Initial guess

$STEP$	x	$f'(x)$	$f(x)$
0	2	3.5838	3.0678
1	$2 - 3.0678/3.5838 = 1.1439$	2.7109	0.3775
2	$1.1439 - 0.3775/2.7109 = 1.004$	2.5451	0.0107

The exact answer is $x^* = 1.0$

Understanding Newton’s Method

- What you want to do is find the root of a function, that is, you are looking for that x^* that makes the function zero: $f(x^*)=0$
- The stumbling block is the nonlinear nature of the function
- Newton’s idea was to linearize the nonlinear function
  - Then search for the root of this new linear function, which hopefully approximates well the original nonlinear function
    - If the only thing that you have is a hammer, then make all your problems a nail

Understanding Newton's Method (Cntd.)

- You are at point q$^{(0)}$ and linearize using Taylor's expansion

$$f(q) = f(q^{(0)}) + \frac{f'(q^{(0)})}{1!}(q - q^{(0)}) + \frac{f''(q^{(0)})}{2!}(q - q^{(0)})^2 + \ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(q^{(0)})}{n!}(q - q^{(0)})^n$$

- After linearization, you end up with linear approximation h(q) of f(q):

$$h(q) = f(q^{(0)}) + \frac{f'(q^{(0)})}{1!}(q - q^{(0)})$$

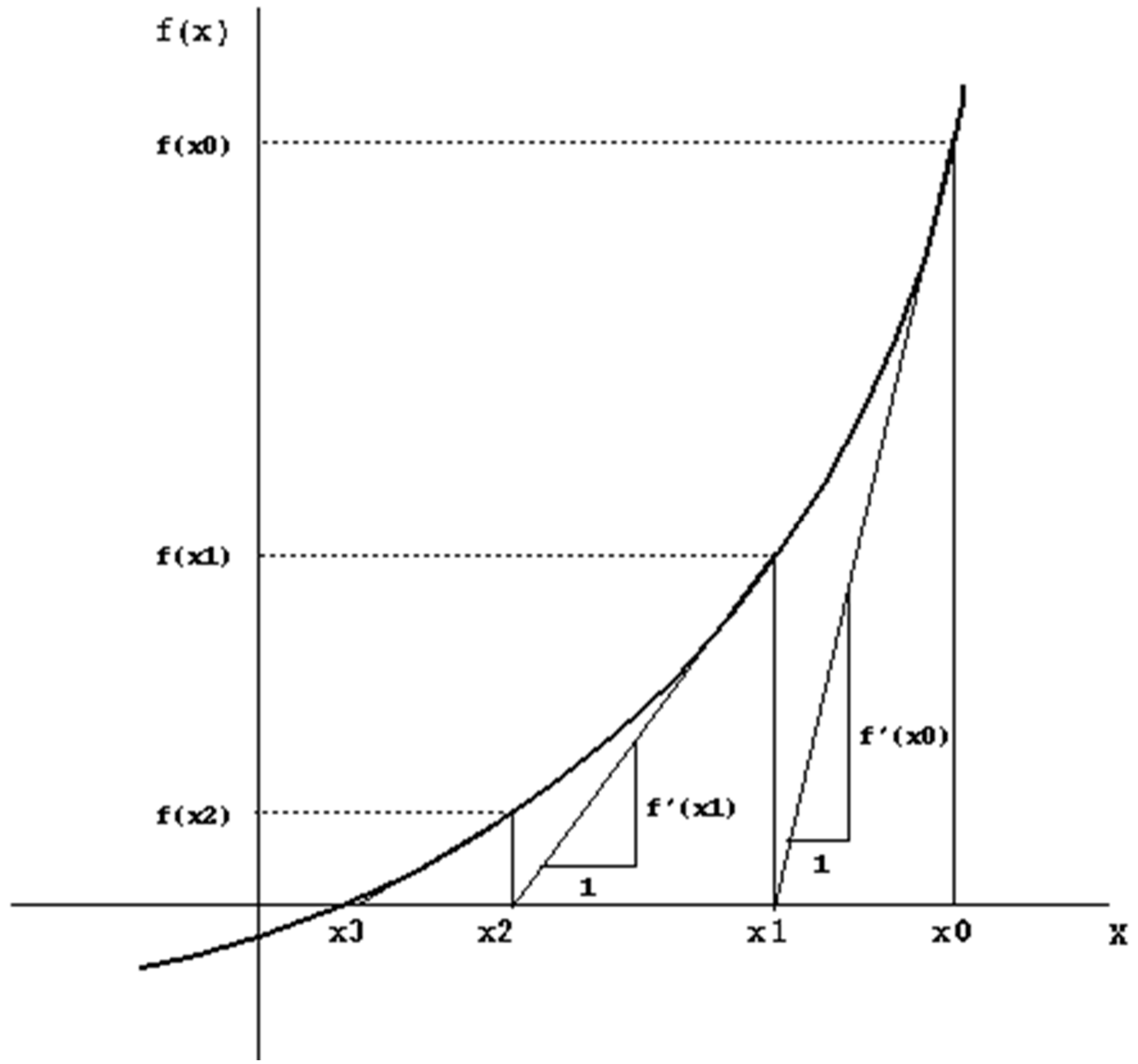
$$\boxed{\text{NOTE:} \quad f(q) \approx h(q)}$$

- Find now q$^{(1)}$ that is the root of h, that is: $h(q^{(1)}) = 0$

$$h(q^{(1)}) = f(q^{(0)}) + \frac{f'(q^{(0)})}{1!}(q^{(1)} - q^{(0)}) = 0 \qquad \Rightarrow \qquad q^{(1)} = q^{(0)} - \frac{f(q^{(0)})}{f'(q^{(0)})}$$

Newton-Raphson Method

Geometric Interpretation



Newton-Raphson
The Multidimensional Case

- Solve for $\mathbf{q} \in \mathbb{R}^n$

the nonlinear system

$$\mathbf{\Phi}(\mathbf{q},t) = \begin{bmatrix} \Phi_1(\mathbf{q},t) \\ \Phi_2(\mathbf{q},t) \\ \vdots \\ \Phi_n(\mathbf{q},t) \end{bmatrix} = \mathbf{0}$$

- The algorithm becomes

$$\mathbf{q}^{(1)} = \mathbf{q}^{(0)} - \left[\mathbf{\Phi}_{\mathbf{q}}(\mathbf{q}^{(0)})\right]^{-1} \cdot \mathbf{\Phi}(\mathbf{q}^{(0)},t)$$

- The Jacobian is defined as

$$\mathbf{\Phi}_{\mathbf{q}}(\mathbf{q}^{(0)}) \equiv \left.\frac{\partial \mathbf{\Phi}}{\partial \mathbf{q}}\right|_{\mathbf{q}=\mathbf{q}^{(0)}}$$

Putting things in perspective…

- Newton algorithm for nonlinear systems requires two things:
  - A starting point $\mathbf{q}^{(0)}$ from where the solution starts being searched for
  - An iterative process in which the approximation of the solution is gradually improved:

$$\mathbf{q}^{(1)} = \mathbf{q}^{(0)} - \left[\boldsymbol{\Phi}_{\mathbf{q}}(\mathbf{q}^{(0)})\right]^{-1} \cdot \boldsymbol{\Phi}(\mathbf{q}^{(0)}, t)$$

$$\mathbf{q}^{(2)} = \mathbf{q}^{(1)} - \left[\boldsymbol{\Phi}_{\mathbf{q}}(\mathbf{q}^{(1)})\right]^{-1} \cdot \boldsymbol{\Phi}(\mathbf{q}^{(1)}, t)$$

$$\mathbf{q}^{(3)} = \mathbf{q}^{(2)} - \left[\boldsymbol{\Phi}_{\mathbf{q}}(\mathbf{q}^{(2)})\right]^{-1} \cdot \boldsymbol{\Phi}(\mathbf{q}^{(2)}, t)$$

$$\cdots \text{etc.}$$

Example: Solve nonlinear system:

$$\begin{cases} x - e^y & = 1 \\ \ln(1+x) - \cos y & = 0 \end{cases}$$

```
% Use Newton method to solve the nonlinear system
% x-exp(y)=0
% log(1+x)-cos(y)=0

% provide initial guess
x = 1;
y = 0;

% start improving the guess
disp('Iteration 1');
residual = [ x-exp(y) ; log(1+x)-cos(y) ];
jacobian = [1  -exp(y) ; 1/(1+x)  sin(y)];
correction = jacobian\residual;
x = x - correction(1)
y = y - correction(2)

disp('Iteration 2');
residual = [ x-exp(y) ; log(1+x)-cos(y) ];
jacobian = [1  -exp(y) ; 1/(1+x)  sin(y)];
correction = jacobian\residual;
x = x - correction(1)
y = y - correction(2)
```

```
Initial guess:
x =  1
y =  0

Iteration 1
x =  1.6137
y =  0.6137

Iteration 2
x =  1.5123
y =  0.4324

Iteration 3
x =  1.5042
y =  0.4085

Iteration 4
x =  1.5040
y =  0.4081

Iteration 5
x =  1.5040
y =  0.4081
```

$$\boxed{\text{Residual:} \quad 10^{-6} \begin{bmatrix} -0.1375 \\ 0.0804 \end{bmatrix}}$$

Newton-Raphson: ME451 Use – Position Analysis of Mechanism

- Problem 3.5.6, modeled with reduced set of generalized coordinates (see handout)

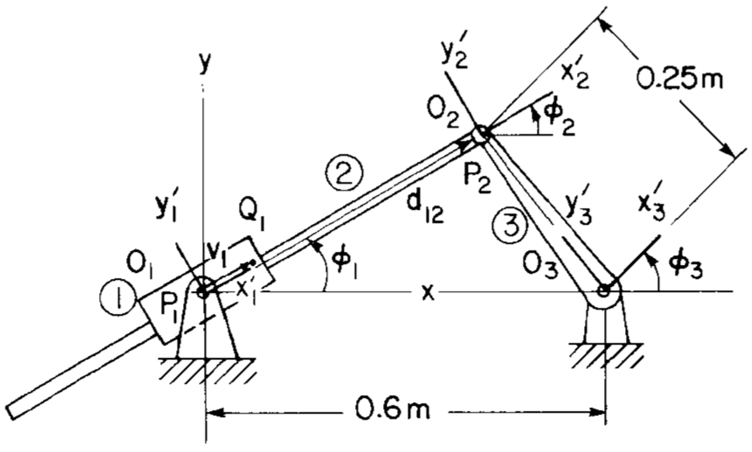


Figure P3.5.6

$$\Phi(\mathbf{q},t)=\begin{bmatrix} x_2-0.6+0.25\sin\phi_3 \\ y_2-0.25\cos\phi_3 \\ x_2^2+y_2^2-(\frac{t}{10}+0.4)^2 \end{bmatrix}=\mathbf{0}$$

$$\mathbf{\Phi_q}(\mathbf{q})=\begin{bmatrix} 1 & 0 & 0.25\cos\phi_3 \\ 0 & 1 & 0.25\sin\phi_3 \\ 2x_2 & 2y_2 & 0 \end{bmatrix}$$

- Iterations carried out like:

$$\mathbf{q}^{(k+1)}=\mathbf{q}^{(k)}-\left[\mathbf{\Phi_q}(\mathbf{q}^{(k)},t)\right]^{-1}\cdot\mathbf{\Phi}(\mathbf{q}^{(k)},t)$$

Dumb approach, which nonetheless works…

```
% Problem 3.5.6
% Solves the nonlinear system associated with Position Analysis
% Numerical solution found using Newton's method.

% get a starting point, that is, an initial guess
q = [0.2 ; 0.2 ; pi/4]

time = 0;
% start improving the guess
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('Iteration 1');
dummy1 = q(1) - 0.6 + 0.25*sin(q(3));
dummy2 = q(2) - 0.25*cos(q(3));
dummy3 = q(1)*q(1) + q(2)*q(2) - (time/10+0.4)*(time/10+0.4);
residual = [ dummy1 ; dummy2 ; dummy3 ];
jacobian = [1  0  0.25*cos(q(3)) ;
   0  1  0.25*sin(q(3));
   2*q(1)  2*q(2)  0];
correction = jacobian\residual;
q = q - correction
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('Iteration 2');
dummy1 = q(1) - 0.6 + 0.25*sin(q(3));
dummy2 = q(2) - 0.25*cos(q(3));
dummy3 = q(1)*q(1) + q(2)*q(2) - (time/10+0.4)*(time/10+0.4);
residual = [ dummy1 ; dummy2 ; dummy3 ];
jacobian = [1  0  0.25*cos(q(3)) ;
   0  1  0.25*sin(q(3));
   2*q(1)  2*q(2)  0];
correction = jacobian\residual;
q = q – correction
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('Iteration 3');
dummy1 = q(1) - 0.6 + 0.25*sin(q(3));
dummy2 = q(2) - 0.25*cos(q(3));
dummy3 = q(1)*q(1) + q(2)*q(2) - (time/10+0.4)*(time/10+0.4);
residual = [ dummy1 ; dummy2 ; dummy3 ];
jacobian = [1  0  0.25*cos(q(3)) ;
   0  1  0.25*sin(q(3));
   2*q(1)  2*q(2)  0];
correction = jacobian\residual;
q = q - correction
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('Iteration 4');
dummy1 = q(1) - 0.6 + 0.25*sin(q(3));
dummy2 = q(2) - 0.25*cos(q(3));
dummy3 = q(1)*q(1) + q(2)*q(2) - (time/10+0.4)*(time/10+0.4);
residual = [ dummy1 ; dummy2 ; dummy3 ];
jacobian = [1  0  0.25*cos(q(3)) ;
   0  1  0.25*sin(q(3));
   2*q(1)  2*q(2)  0];
correction = jacobian\residual;
q = q - correction
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('Iteration 5');
dummy1 = q(1) - 0.6 + 0.25*sin(q(3));
dummy2 = q(2) - 0.25*cos(q(3));
dummy3 = q(1)*q(1) + q(2)*q(2) - (time/10+0.4)*(time/10+0.4);
residual = [ dummy1 ; dummy2 ; dummy3 ];
jacobian = [1  0  0.25*cos(q(3)) ;
   0  1  0.25*sin(q(3));
   2*q(1)  2*q(2)  0];
correction = jacobian\residual;
q = q - correction
```

Example: Position Analysis of Mechanism

- Output of the code shows how the position configuration at time t=0 is gradually improved

```
Initial guess: q =   0.2000   0.2000   0.7854
 Iteration 1:  q =   0.4232   0.1768   0.7854
 Iteration 2:  q =   0.3812   0.1348   1.0228
 Iteration 3:  q =   0.3813   0.1216   1.0635
 Iteration 4:  q =   0.3813   0.1210   1.0654
 Iteration 5:  q =   0.3813   0.1210   1.0654
```

- After 4 iterations it doesn’t make sense to keep iterating, the solution is already very good

Better way to implement Position Analysis…

```
% Problem 3.5.6
% Solves the nonlinear system associated with Position Analysis
% Numerical solution found using Newton's method.


% get a starting point, that is, an initial guess
q = [0.2 ; 0.2 ; pi/4]

time = 0;
% start improving the guess
for i = 1:6
    crnt_iteration = strcat('Iteration ', int2str(i));
    disp(crnt_iteration);
    dummy1 = q(1) - 0.6 + 0.25*sin(q(3));
    dummy2 = q(2) - 0.25*cos(q(3));
    dummy3 = q(1)*q(1) + q(2)*q(2) - (time/10+0.4)*(time/10+0.4);
    residual = [ dummy1 ; dummy2 ; dummy3 ];
    jacobian = [1  0  0.25*cos(q(3)) ;
        0  1  0.25*sin(q(3));
        2*q(1)  2*q(2)  0];
    correction = jacobian\residual;
    q = q - correction
end
```

This way is better since you introduced an iteration loop and go through six iterations before you stop. Compared to previous solution it saves a lot of coding effort and makes it more clear and compact

Even better way to implement Position Analysis…

```
% Problem 3.5.6
% Solves the nonlinear system associated with Position Analysis
% Numerical solution found using Newton's method.

% get a starting point, that is, an initial guess
q = [0.2 ; 0.2 ; pi/4]

time = 0;
% start improving the guess
normCorrection = 1.0;
tolerance = 1.E-8;
iterationCounter = 0;
while normCorrection>tolerance,
   iterationCounter = iterationCounter + 1;
   crnt_iteration = strcat('Iteration ', int2str(iterationCounter));
   disp(crnt_iteration);
   dummy1 = q(1) - 0.6 + 0.25*sin(q(3));
   dummy2 = q(2) - 0.25*cos(q(3));
   dummy3 = q(1)*q(1) + q(2)*q(2) - (time/10+0.4)*(time/10+0.4);
   residual = [ dummy1 ; dummy2 ; dummy3 ];
   jacobian = [1  0  0.25*cos(q(3)) ;
      0  1  0.25*sin(q(3));
      2*q(1)  2*q(2)  0];
   correction = jacobian\residual;
   q = q - correction;
   normCorrection = norm(correction);
end
disp('Here is the value of q:'), q
```

This way is even better since you keep iterating until the norm of the correction becomes very small…

Position Analysis: making it general…

```
% Problem 3.5.6
% Solves nonlinear system associated with Position Analysis
% Numerical solution found using Newton's method.

% get a starting point, that is, an initial guess
q = [0.2 ; 0.2 ; pi/4]

time = 0;
% start improving the guess
normCorrection = 1.0;
tolerance = 1.E-8;
iterationCounter = 1;
while normCorrection>tolerance,
   residual = getPhi(q, time);
   jacobian = getJacobian(q);
   correction = jacobian\residual;
   q = q - correction;
   normCorrection = norm(correction);
   iterationCounter = iterationCounter + 1;
end
disp('Here is the value of q:'), q
```

```
function phi = getPhi(q, t)
% computes the violation in constraints
% Input : current position q, and current time t
% Output: returns constraint violations

dummy1 = q(1) - 0.6 + 0.25*sin(q(3));
dummy2 = q(2) - 0.25*cos(q(3));
dummy3 = q(1)*q(1) + q(2)*q(2) - (t/10+0.4)*(t/10+0.4);
phi = [ dummy1 ; dummy2 ; dummy3 ];
```

```
function jacobian = getJacobian(q)
% computes the Jacobian of the  constraints
% Input : current position q
% Output: Jacobian matrix J

jacobian = [   1  0  0.25*cos(q(3)) ;
   0  1  0.25*sin(q(3));
   2*q(1)  2*q(2)  0    ];
```

Newton’s Method: Closing Remarks

- Can ever things go wrong with Newton’s method?
- Yes, there are at least three instances:

1. Most commonly, the starting point is not close to the solution that you try to find and the iterative algorithm diverges (goes to infinity)
2. Since a nonlinear system can have multiple solutions, the Newton algorithm finds a solution that is not the one sought (happens if you don’t choose the starting point right)
3. The speed of convergence of the algorithm is not good (happens if the Jacobian is close to being singular (zero determinant) at the root, not that common)

Newton's Method: Closing Remarks

- What can you do address these issues?
- You cannot do anything about 3 above, but can fix 1 and 2 provided you choose your starting point carefully
- Newton's method converges very fast (quadratically) if started close enough to the solution
- To help Newton's method in Position Analysis, you can take the starting point of the algorithm at time t_k to be the value of $\mathbf{q}$ from t_{k-1} (that is, the very previous configuration of the mechanism)

Position Analysis: Final Form

```
% Driver for Position Analysis
% Works for kinematic analysis of any mechanism
% User needs to implement three subroutines:
% provideInitialGuess
% getPhi
% getJacobian

% general settings
normCorrection = 1.0;
tolerance = 1.E-8;
timeStep = 0.05;
timeEnd = 4;
timePoints = 0:timeStep:timeEnd;
results = zeros(3,length(timePoints));
crntOutputPoint = 0;

% start the solution loop
q = provideInitialGuess();
for time = 0:timeStep:timeEnd
   crntOutputPoint = crntOutputPoint + 1;
   while normCorrection>tolerance,
      residual = getPhi(q, time);
      jacobian = getJacobian(q);
      correction = jacobian\residual;
      q = q - correction;
      normCorrection = norm(correction);
   end
   results(:, crntOutputPoint) = q;
   normCorrection = 1;
end
```

```
function q_zero = provideInitialGuess()
% purpose of function is to provide an initial
% guess to start the Newton algorithm at
% time=0

q_zero = [0.2 ; 0.2 ; pi/4];
```

```
function phi = getPhi(q, t)
% computes the violation in constraints
% Input : current position q, and current time t
% Output: returns constraint violations

dummy1 = q(1) - 0.6 + 0.25*sin(q(3));
dummy2 = q(2) - 0.25*cos(q(3));
dummy3 = q(1)*q(1) + q(2)*q(2) - (t/10+0.4)*(t/10+0.4);
phi = [ dummy1 ; dummy2 ; dummy3 ];
```

```
function jacobian = getJacobian(q)
% computes the Jacobian of the  constraints
% Input : current position q
% Output: Jacobian matrix J

jacobian = [   1  0  0.25*cos(q(3)) ;
   0  1  0.25*sin(q(3));
   2*q(1)  2*q(2)  0    ];
```

END: Newton Method

Begin: Closing Remarks, Kinematics

Mechanism Analysis: Steps

- **Step A**: Identify *all* physical joints and drivers present in the system
- **Step B**: Identify the corresponding set of constraint equations $\Phi(\mathbf{q},t)=0$
- **Step C**: Solve for the Position as a function of time ($\Phi_{\mathbf{q}}$ is needed)
- **Step D**: Solve for the Velocities as a function of time (ν is needed)
- **Step E**: Solve for the Accelerations as a function of time (γ is needed)

Position, Velocity, and Acceleration Analysis (Section 3.6)

- The position analysis [Step C]:
  - It's the tougher of the three
  - Requires the solution of a system of nonlinear equations
  - What you are after is determining at each time the location and orientation of each component (body) of the mechanism

- The velocity analysis [Step D]:
  - Requires the solution of a linear system of equations
  - Relatively simple
  - Carried out after you are finished with the position analysis

- The acceleration analysis [Step E]:
  - Requires the solution of a linear system of equations
  - What is challenging is generating the RHS of acceleration equation, γ
  - Carried out after you are finished with the velocity analysis

Position Analysis

- Framework:
  - Somebody presents you with a mechanism and you select the set of *nc* generalized coordinates to position and orient each body of the mechanism:

$$\mathbf{q} = [x_1, y_1, \phi_1, x_2, y_2, \phi_2, \ldots]^T \in \mathbb{R}^{nc}$$

  - You inspect the mechanism and identify a set of *nk* kinematic constraints that must be satisfied by your coordinates:

$$\Phi^K(\mathbf{q}) = \mathbf{0}$$

  - Next, you identify the set of *nd* driver constraints that move the mechanism:

$$\Phi^D(\mathbf{q}, \mathbf{t}) = \mathbf{0}$$

NOTE: YOU MUST HAVE *nc = nk + nd*

Position Analysis

- We end up with this problem: given a time value *t*, find that set of generalized coordinates **q** that satisfy the equations:

$$\Phi(\mathbf{q}, t) = \begin{bmatrix} \Phi^K(\mathbf{q}) \\ \Phi^D(\mathbf{q}, t) \end{bmatrix} = \mathbf{0}$$

- What's the idea here?
  - Set time t=0, and find a solution **q** by solving above equations with Newton-Raphson
  - Then advance the time to t=0.001 and find a solution **q** by solving above equations
  - Then advance the time to t=0.002 and find a solution **q** by solving above equations
  - Then advance the time to t=0.003 and find a solution **q** by solving above equations
  - …
  - Stop when you reach the end of the interval in which you are interested in the position

- What you do is find the time evolution on a **time grid** with step size Δt=0.001
  - You can then plot the solution as a function of time and get the time evolution of your mechanism

Position Analysis

- Two issues associated with the methodology described on previous slide:
  - The first issue: related to the fact that you are solving nonlinear equations.
    - Does a solution exist? Example: $x^2+4=0$ (no real number x will do here)
    - Is the solution unique? Example: $x^2-4=0$x (both 2 and -2 are ok solutions)
  - The second issue: The equations that we have to solve at each time *t* are nonlinear.
    - For instance, how do you find the solution (x=-1.2) of this nonlinear equation:

$$[\ln(\cos(x)) + 1]^2 - \sin(x^2 + 1) + 0.645206284641418 = 0$$

    - This is why the Newton-Raphson method was needed

Position Analysis: Implicit Function Theorem

- Is the solution of our nonlinear system well behaved? That is, does it exist, and is it unique?
  - A sufficient condition is provided by the **Implicit Function Theorem**

- In layman's words, this is what the Implicit Function Theorem says:
  - Assume that we are at some time t_k, and we just found the solution $\mathbf{q}_{(k)}$ and we question the quality of this solution
  - If the constraint Jacobian is nonsingular in this configuration, that is,

$$\det \left| \mathbf{\Phi}_{\mathbf{q}}(\mathbf{q}_{(k)}, t_k) \right| \neq 0$$

  - … then, we can conclude that the solution is unique, and not only at t_k, but in a small interval δ about time t_k.
  - Additionally, in this small time interval, there is an explicit functional dependency of **q** on t, that is, there is a function **f**(t) such that:

$$\mathbf{q}(t) = \mathbf{f}(t) \quad \text{for} \quad |t - t_k| < \delta$$

Quick Comments…

- What does it mean "the solution is unique, and not only at t_k, but in a small interval δ about time t_k"?
  - It means that if you look back in time a little bit (a value δ), or if you look ahead in time a little bit (a value δ), the mechanism in this time interval is guaranteed to be "healthy" (the constraint equations are well defined and around t_k the mechanism assumes a unique configuration)

- Notation used on previous slide: note that the subscript is in parentheses

$$\mathbf{q}_{(k)}, \quad \dot{\mathbf{q}}_{(k)}$$

  - It indicates that that quantity is evaluated at t_k
  - If no parentheses, can be mistaken for the coordinates associated with body "k"

End Position Analysis

Begin Velocity Analysis

Velocity Analysis

- This is simple. What is the framework?
- You just found **q** at time *t*, that is, the location and orientation of each component of the mechanism at time t, and now you want to find the velocity of each component (body) of the mechanism
- Taking one time derivative of the constraints leads to the velocity equation:

$$\mathbf{\Phi}(\mathbf{q},t)=\mathbf{0} \quad \Rightarrow \quad \dot{\mathbf{\Phi}}(\mathbf{q},t)=\mathbf{0} \quad \Leftrightarrow \quad \mathbf{\Phi}_{\mathbf{q}}(\mathbf{q},t)\cdot\dot{\mathbf{q}}=\nu$$

- In layman's words, once you have $\mathbf{q}_{(k)}$ you can find $\dot{\mathbf{q}}_{(k)}$ at time t_k by solving the linear system

$$\mathbf{\Phi}_{\mathbf{q}}(\mathbf{q}_{(k)},t_k)\cdot\dot{\mathbf{q}}_{(k)}=\nu_{(k)}$$

Velocity Analysis

- Observations:
  - Note that as long as the constraint Jacobian is nonsingular, you can solve this linear system and recover the velocity $\dot{\mathbf{q}}_{(k)}$
  - The reason velocity analysis is easy is that, unlike for position analysis where you have to solve a nonlinear system, now you are dealing with a linear system, which is easy to solve
  - Note that the velocity analysis comes after the position analysis is completed, and you are in a new configuration of the mechanism in which you are about to find out its velocity

End Velocity Analysis

Begin Acceleration Analysis

Acceleration Analysis

- This is also fairly simple. What is the framework?
- You just found $\mathbf{q}_{(k)}$ and $\dot{\mathbf{q}}_{(k)}$ at time t_k, that is, where the mechanism is at time t_k, and what its velocity is
- You'd like to know the acceleration of each component of the model
- Taking two time derivatives of the constraints leads to the acceleration equation:

$$\mathbf{\Phi}(\mathbf{q}_{(k)}, t_k) = \mathbf{0} \quad \Rightarrow \quad \ddot{\mathbf{\Phi}}(\mathbf{q}_{(k)}, t_k) = \mathbf{0} \quad \Leftrightarrow \quad \mathbf{\Phi}_{\mathbf{q}}(\mathbf{q}_{(k)}, t_k) \cdot \ddot{\mathbf{q}}_{(k)} = \gamma_{(k)}$$

Acceleration Analysis

- In other words, you find the acceleration (second time derivative of **q** at time t_k) as the solution of a linear system:

$$\mathbf{\Phi}_{\mathbf{q}}(\mathbf{q}_{(k)}, t_k) \cdot \ddot{\mathbf{q}}_{(k)} = \gamma_{(k)}$$

- Observations:
  - The equation above illustrates why we have been interested in the expression of γ, the RHS of the acceleration equation:

$$\gamma = -(\mathbf{\Phi}_{\mathbf{q}}\dot{\mathbf{q}})_{\mathbf{q}}\dot{\mathbf{q}} - 2\mathbf{\Phi}_{\mathbf{q}}t\dot{\mathbf{q}} - \mathbf{\Phi}_{tt}$$

  - Note that you again want the constraint Jacobian to be nonsingular, since then you can solve the acceleration linear system and obtained the acceleration $\ddot{\mathbf{q}}_{(k)}$

SUMMARY OF CHAPTER 3

- We looked at the KINEMATICS of a mechanism
  - That is, we are interested in how this mechanism moves in response to a set of kinematic drives (motions) applied to it

- What one has to do:
  - **Step A**: Identify *all* physical *joints* and *drivers* present in the system
  - **Step B**: Identify the corresponding set of constraint equations $\Phi(\mathbf{q},t)=0$
  - **Step C**: Solve for the Position as a function of time ($\Phi_{\mathbf{q}}$ is needed)
  - **Step D**: Solve for the Velocities as a function of time (ν is needed)
  - **Step E**: Solve for the Accelerations as a function of time (γ is needed)