ME451
Kinematics and Dynamics of Machine Systems

Start Position, Velocity, and Acc. Analysis 3.6
October 13, 2011

"The best way to predict the future is to invent it."
Alan Kay

Before we get started…

- Last Time
  - Discussed driving constraints
    - To carry out Kinematics Analysis you need to have zero net degrees of freedom
      - NDOF=0
    - Driver constraints
      - Absolute: x, y, ϕ
      - Relative: distance, motion on a revolute joint, motion on a translational joint

- Today:
  - Excavator example, setting up driving constraints
  - Kinematic Analysis Wrap-up: Position, Velocity, and Acceleration Stages

- Midterm exam coming up on November 3

- Assignment due on Th, Oct. 20:
  - Textbook: 3.4.7, 3.4.8, 3.4.9
  - ADAMS & MATLAB emailed to you

Example: Specifying Relative Distance Drivers

- Generalized coordinates: $\mathbf{q} = [\phi_1, x_2, y_2, \phi_2]^T$

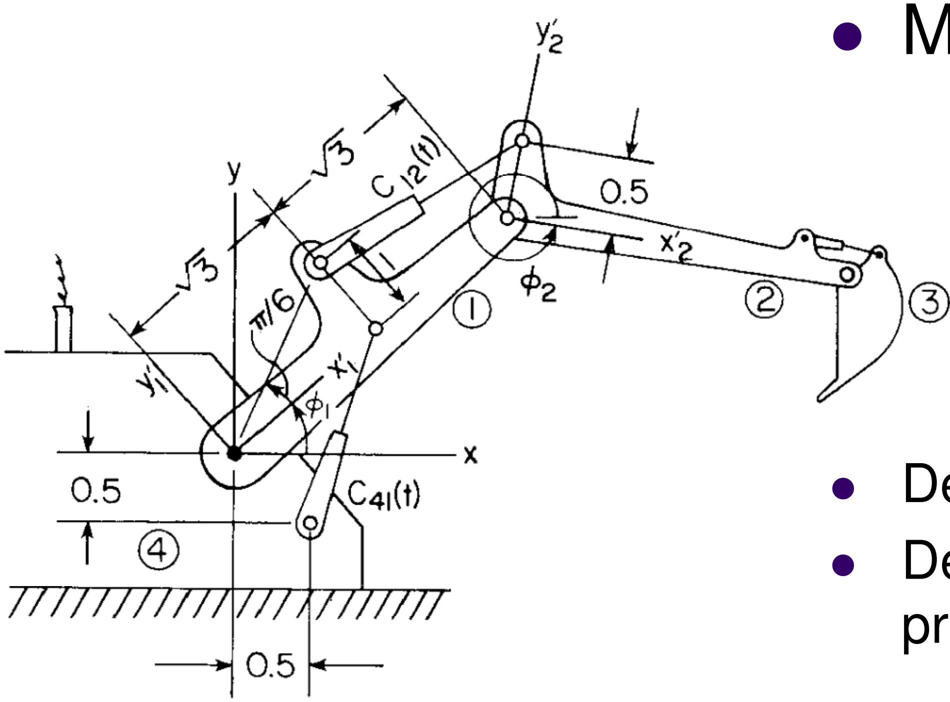

Figure 3.5.6 Excavator boom assembly with two distance drivers.

- Motions prescribed:

$$C_{41}(t) = \frac{1}{5}\,t + 1.8$$

$$C_{12}(t) = \frac{1}{10}\,t + 1.9$$

- Derive the constraints acting on system
- Derive linear system whose solution provides velocities $\dot{\mathbf{q}} = [\dot{\phi}_1, \dot{x}_2, \dot{y}_2, \dot{\phi}_2]^T$

Mechanism Analysis: Steps

- **Step A**: Identify *all* physical joints and drivers present in the system
- **Step B**: Identify the corresponding set of constraint equations $\Phi(\mathbf{q},t)=0$
- **Step C**: Solve for the Position as a function of time ($\Phi_{\mathbf{q}}$ is needed)
- **Step D**: Solve for the Velocities as a function of time (ν is needed)
- **Step E**: Solve for the Accelerations as a function of time (γ is needed)

Position, Velocity, and Acceleration Analysis (Section 3.6)

- The position analysis [Step C]:
  - It's the tougher of the three
  - Requires the solution of a system of nonlinear equations
  - What you are after is determining at each time the location and orientation of each component (body) of the mechanism

- The velocity analysis [Step D]:
  - Requires the solution of a linear system of equations
  - Relatively simple
  - Carried out after you are finished with the position analysis

- The acceleration analysis [Step E]:
  - Requires the solution of a linear system of equations
  - What is challenging is generating the RHS of acceleration equation, γ
  - Carried out after you are finished with the velocity analysis

Position Analysis

- Framework:
  - Somebody presents you with a mechanism and you select the set of *nc* generalized coordinates to position and orient each body of the mechanism:

$$\mathbf{q} = [x_1, y_1, \phi_1, x_2, y_2, \phi_2, \ldots]^T \in \mathbb{R}^{nc}$$

  - You inspect the mechanism and identify a set of *nk* kinematic constraints that must be satisfied by your coordinates:

$$\Phi^K(\mathbf{q}) = \mathbf{0}$$

  - Next, you identify the set of *nd* driver constraints that move the mechanism:

$$\Phi^D(\mathbf{q}, \mathbf{t}) = \mathbf{0}$$

NOTE: YOU MUST HAVE *nc* = *nk* + *nd*

Position Analysis

- We end up with this problem: given a time value *t*, find that set of generalized coordinates **q** that satisfy the equations:

$$\Phi(\mathbf{q}, t) = \begin{bmatrix} \Phi^K(\mathbf{q}) \\ \Phi^D(\mathbf{q}, t) \end{bmatrix} = \mathbf{0}$$

- What's the idea here?
  - Set time t=0, and find a solution **q** by solving above equations
  - Then advance the time to t=0.001 and find a solution **q** by solving above equations
  - Then advance the time to t=0.002 and find a solution **q** by solving above equations
  - Then advance the time to t=0.003 and find a solution **q** by solving above equations
  - …
  - Stop when you reach the end of the interval in which you are interested in the position

- What you do is find the time evolution on a **time grid** with step size $\Delta t=0.001$
  - You can then plot the solution as a function of time and get the time evolution of your mechanism

Position Analysis

- Two issues associated with the methodology described on previous slide:
  - The first issue: related to the fact that you are solving nonlinear equations.
    - Does a solution exist? Example: $x^2+4=0$ (no real number x will do here)
    - Is the solution unique? Example: $x^2-4=0x$ (both 2 and -2 are ok solutions)
  - The second issue: The equations that we have to solve at each time *t* are nonlinear. How do you actually solve them?
    - For instance, how do you find the solution (x=-1.2) of this nonlinear equation:

$$[\ln(\cos(x)) + 1]^2 - \sin(x^2 + 1) + 0.645206284641418 = 0$$

    - Deal with this issue next week (discuss Newton-Raphson method)

Position Analysis: Implicit Function Theorem

- Is the solution of our nonlinear system well behaved? That is, does it exist, and is it unique?
  - A sufficient condition is provided by the **Implicit Function Theorem**

- In layman's words, this is what the Implicit Function Theorem says:
  - Assume that we are at some time t_k, and we just found the solution $\mathbf{q}_{(k)}$ and we question the quality of this solution
  - If the constraint Jacobian is nonsingular in this configuration, that is,

$$\det |\mathbf{\Phi}_{\mathbf{q}}(\mathbf{q}_{(k)}, t_k)| \neq 0$$

  - … then, we can conclude that the solution is unique, and not only at t_k, but in a small interval δ about time t_k.
  - Additionally, in this small time interval, there is an explicit functional dependency of **q** on t, that is, there is a function **f**(t) such that:

$$\mathbf{q}(t) = \mathbf{f}(t) \quad \text{for} \quad |t - t_k| < \delta$$

Quick Comments…

- What does it mean "the solution is unique, and not only at t_k, but in a small interval δ about time t_k"?
  - It means that if you look back in time a little bit (a value δ), or if you look ahead in time a little bit (a value δ), the mechanism during this period is guaranteed to be "healthy" (the constraint equations are well defined and around t_k the mechanism assumes a unique configuration)

- Notation used on previous slide: note that the subscript is in parentheses

$$\mathbf{q}_{(k)}, \quad \dot{\mathbf{q}}_{(k)}$$

  - It indicates that that quantity is evaluated at t_k
  - If no parentheses, can be mistaken for the coordinates associated with body "k"

End Position Analysis

Begin Velocity Analysis

Velocity Analysis

- This is simple. What is the framework?

- You just found **q** at time *t*, that is, the location and orientation of each component of the mechanism at time t, and now you want to find the velocity of each component (body) of the mechanism

- Taking one time derivative of the constraints leads to the velocity equation:

$$\mathbf{\Phi}(\mathbf{q},t)=\mathbf{0} \quad \Rightarrow \quad \dot{\mathbf{\Phi}}(\mathbf{q},t)=\mathbf{0} \quad \Leftrightarrow \quad \mathbf{\Phi}_{\mathbf{q}}(\mathbf{q},t)\cdot\dot{\mathbf{q}}=\nu$$

- In layman's words, once you have $\mathbf{q}_{(k)}$ you can find $\dot{\mathbf{q}}_{(k)}$ at time t_k by solving the linear system

$$\mathbf{\Phi}_{\mathbf{q}}(\mathbf{q}_{(k)},t_k)\cdot\dot{\mathbf{q}}_{(k)}=\nu_{(k)}$$

Velocity Analysis

- Observations:
  - Note that as long as the constraint Jacobian is nonsingular, you can solve this linear system and recover the velocity $\dot{\mathbf{q}}_{(k)}$
  - The reason velocity analysis is easy is that, unlike for position analysis where you have to solve a nonlinear system, now you are dealing with a linear system, which is easy to solve
  - Note that the velocity analysis comes after the position analysis is completed, and you are in a new configuration of the mechanism in which you are about to find out its velocity

End Velocity Analysis

Begin Acceleration Analysis

Acceleration Analysis

- This is also fairly simple. What is the framework?
- You just found $\mathbf{q}_{(k)}$ and $\dot{\mathbf{q}}_{(k)}$ at time t_k, that is, where the mechanism is at time t_k, and what its velocity is
- You'd like to know the acceleration of each component of the model
- Taking two time derivatives of the constraints leads to the acceleration equation:

$$\mathbf{\Phi}(\mathbf{q}_{(k)}, t_k) = \mathbf{0} \quad \Rightarrow \quad \ddot{\mathbf{\Phi}}(\mathbf{q}_{(k)}, t_k) = \mathbf{0} \quad \Leftrightarrow \quad \mathbf{\Phi}_{\mathbf{q}}(\mathbf{q}_{(k)}, t_k) \cdot \ddot{\mathbf{q}}_{(k)} = \gamma_{(k)}$$

Acceleration Analysis

- In other words, you find the acceleration (second time derivative of **q** at time t_k) as the solution of a linear system:

$$\mathbf{\Phi_q}(\mathbf{q}_{(k)}, t_k) \cdot \ddot{\mathbf{q}}_{(k)} = \gamma_{(k)}$$

- Observations:
  - The equation above illustrates why we have been interested in the expression of γ, the RHS of the acceleration equation:

$$\gamma = -(\mathbf{\Phi_q}\dot{\mathbf{q}})_{\mathbf{q}}\dot{\mathbf{q}} - 2\mathbf{\Phi_q}t\dot{\mathbf{q}} - \mathbf{\Phi}_{tt}$$

  - Note that you again want the constraint Jacobian to be nonsingular, since then you can solve the acceleration linear system and obtained the acceleration $\ddot{\mathbf{q}}_{(k)}$

SUMMARY OF CHAPTER 3

- We looked at the KINEMATICS of a mechanism
  - That is, we are interested in how this mechanism moves in response to a set of kinematic drives (motions) applied to it

- What one has to do:
  - **Step A**: Identify *all* physical *joints* and *drivers* present in the system
  - **Step B**: Identify the corresponding set of constraint equations $\Phi(\mathbf{q},t)=0$
  - **Step C**: Solve for the Position as a function of time ($\Phi_{\mathbf{q}}$ is needed)
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