ME451
Kinematics and Dynamics of Machine Systems

Relative Kinematic Constraints, Composite Joints – 3.3

October 4, 2011

"The real problem is not whether machines think but whether men do."
B. F. Skinner

Before we get started…

- Last Time
  - We looked at several absolute constraints
    - x, y, ϕ relative constraints
    - Started absolute distance constraint

  - For each kinematic constraint, recall the drill that you have to go through in order to provide what it takes to carry out Kinematics Analysis
    - Five step procedure:
      - Identify and analyze the physical joint
      - Derive the constraint equations associated with the joint
      - Compute constraint Jacobian $\Phi_{\mathbf{q}}$
      - Get ν (RHS of velocity equation)
      - Get γ (RHS of acceleration equation, this is challenging in some cases)

- Today
  - Covering relative constraints:
    - Revolute, translational, and composite joints

Absolute distance-constraint

- Step 1: the distance from a point P_i to an absolute (or global) reference frame stays constant, and equal to some known value C_4

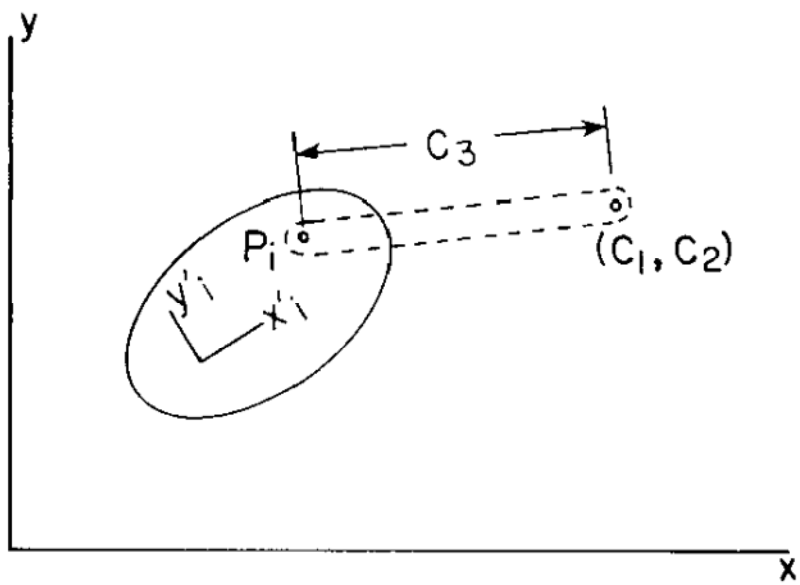


Figure 3.2.1 Constraint that distance from point P to (C_1, C_2) equals C_3.

- Step 2: Identify $\Phi^{dx(i)}=0$
- Step 3: $\Phi^{dx(i)}_{\mathbf{q}} = ?$
- Step 4: $\nu^{dx(i)} = ?$
- Step 5: $\gamma^{dx(i)} = ?$

Example 3.2.1

- An example where you'd have to use absolute constraints: the simple pendulum

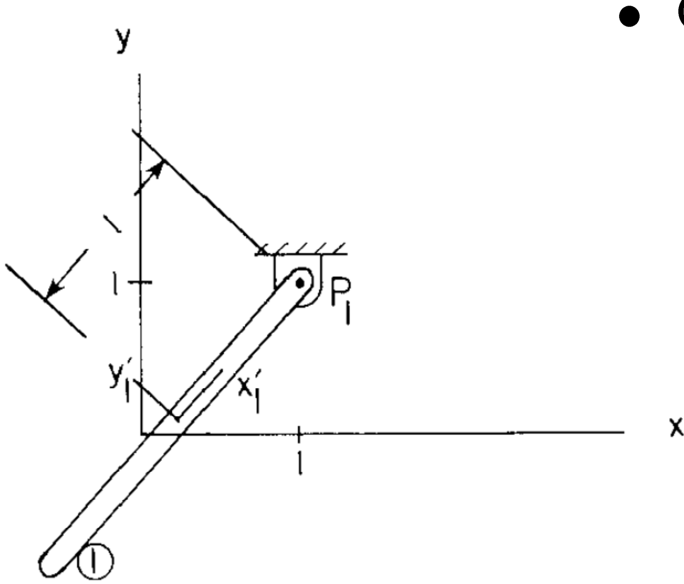


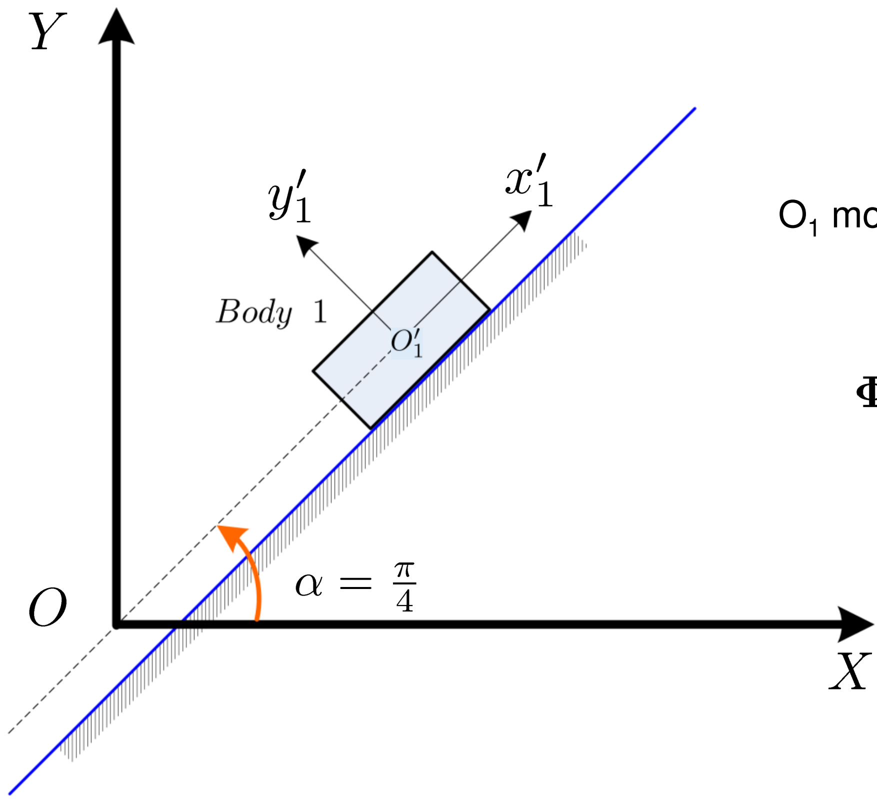
Figure 3.2.3 Simple pendulum with absolute constraints.

- Generalized coordinates used:

$$\mathbf{q}_1 = \begin{bmatrix} x_1 \\ y_1 \\ \phi_1 \end{bmatrix}$$

Example 3.1.3 [Tricky]

- A case when the algebraic constraints fail to imply (enforce) the actual kinematics of a mechanism: block sliding on incline, incline angle π/4
  - Use the following set of generalized coordinates: $\mathbf{q} = \begin{bmatrix} x_1 \\ y_1 \\ \varphi_1 \end{bmatrix}$
  - Formulate constraints defining the kinematics (motion) of the mechanism



O_1 moves along $\alpha = \pi/4$

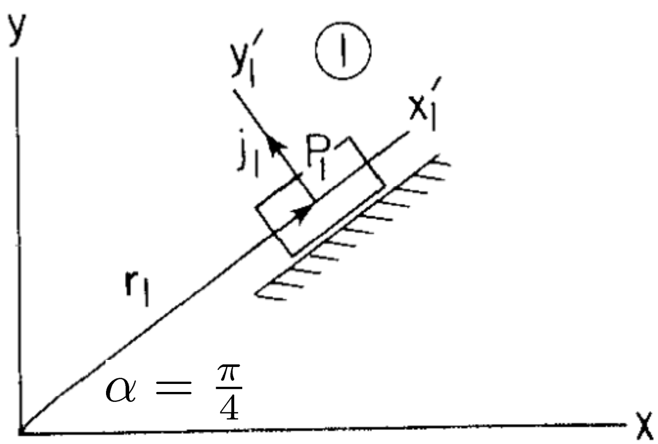
$\overrightarrow{OO_1'} \cdot \overrightarrow{\mathbf{j}_1'} = 0$

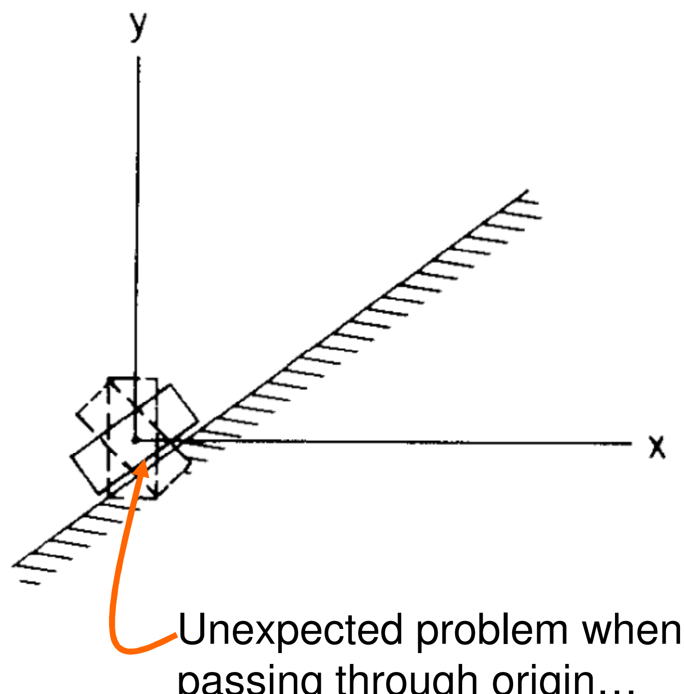
$$\mathbf{\Phi}(\mathbf{q}, t) = \begin{bmatrix} x_1 - y_1 \\ -x_1 \sin\varphi_1 + y_1 \cos\varphi_1 \\ x_1 + 6 - 6t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

User prescribed motion (given)

Example 3.1.3

- Note that when passing through the origin, the algebraic constraints fail to specify the actual kinematics of the mechanism
- An example when the translating plain English into equations is not straightforward





- Translating plain English into the right equations:

$$\mathbf{\Phi}(\mathbf{q},t) = \begin{bmatrix} x_1 - y_1 \\ \varphi_1 - \pi/4 \\ x_1 + 6 - 6t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Example 3.2.2

- An example where you'd have to use an absolute angle constraint: slider along x-axis

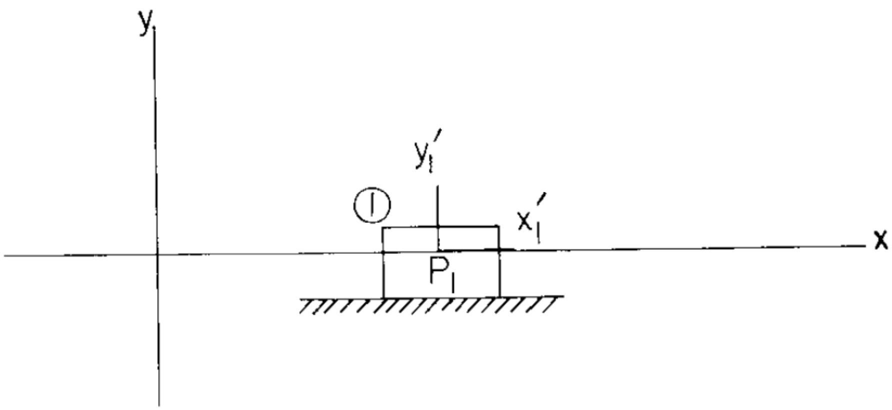


Figure 3.2.4 Slider along x axis.

Moving on to <u>relative</u> constraints
(section 3.3)

Loose Ends:

Switching representation between two Reference Frames with rotation matrices A_i and A_j, respectively

- The problem at hand:
  - The representation of a geometric vector in a RF with $\mathbf{A}_j$ (described by angle ϕ_j) is $\bar{\mathbf{s}}_j^P$
  - Now you want to determine the representation $\bar{\mathbf{s}}_i^P$ of the same geometric vector in a RF with $\mathbf{A}_i$ (described by angle ϕ_i)
  - How do you do this?

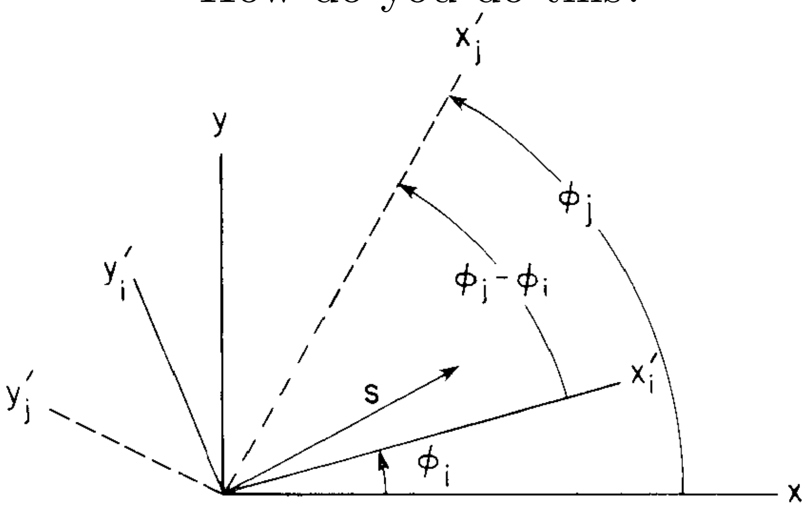


Figure 2.4.4 Three reference frames with coincident origins.

$$\bar{\mathbf{s}}_i^P = \mathbf{A}_i^T \mathbf{A}_j \bar{\mathbf{s}}_j^P$$

Recall our notation: $\mathbf{A}(\phi_i) \equiv \mathbf{A}_i$

Loose Ends, Continued:

Notation (related to changing representation from $\mathbf{A}_j$ to $\mathbf{A}_i$)

- Note that: $$\mathbf{A}_i^T \mathbf{A}_j = \begin{bmatrix} \cos\phi_i & \sin\phi_i \\ -\sin\phi_i & \cos\phi_i \end{bmatrix} \cdot \begin{bmatrix} \cos\phi_j & -\sin\phi_j \\ \sin\phi_j & \cos\phi_j \end{bmatrix}$$

$$\Downarrow$$

$$\mathbf{A}_i^T \mathbf{A}_j = \begin{bmatrix} \cos(\phi_j - \phi_i) & -\sin(\phi_j - \phi_i) \\ \sin(\phi_j - \phi_i) & \cos(\phi_j - \phi_i) \end{bmatrix}$$

- Then, using for the angle between the two reference frames the notation

$$\phi_{ij} \equiv \phi_j - \phi_i \quad ,$$

- … we get that

$$\mathbf{A}_i^T \mathbf{A}_j = \mathbf{A}(\phi_j - \phi_i) = \mathbf{A}(\phi_{ij}) \equiv \mathbf{A}_{ij} \qquad \Rightarrow \qquad \bar{\mathbf{s}}_i^P = \mathbf{A}_{ij} \bar{\mathbf{s}}_j^P$$

- Note that the order is important: it is "$\mathbf{A}_{ij}$" and not "$\mathbf{A}_{ji}$"

- $\mathbf{A}_{ij}$ gets multiplied from the right by a vector represented in the "j" Reference Frame (RF) and produces a vector represented in the "i" RF

- Note that when you see $\mathbf{A}_j$ in fact you should had $\mathbf{A}_{0j}$, where "0" is used to symbolize the global reference frame

Loose Ends, Final Slide

(related to changing representation from A_j to A_i)

- For later reference, it is useful to recall that,

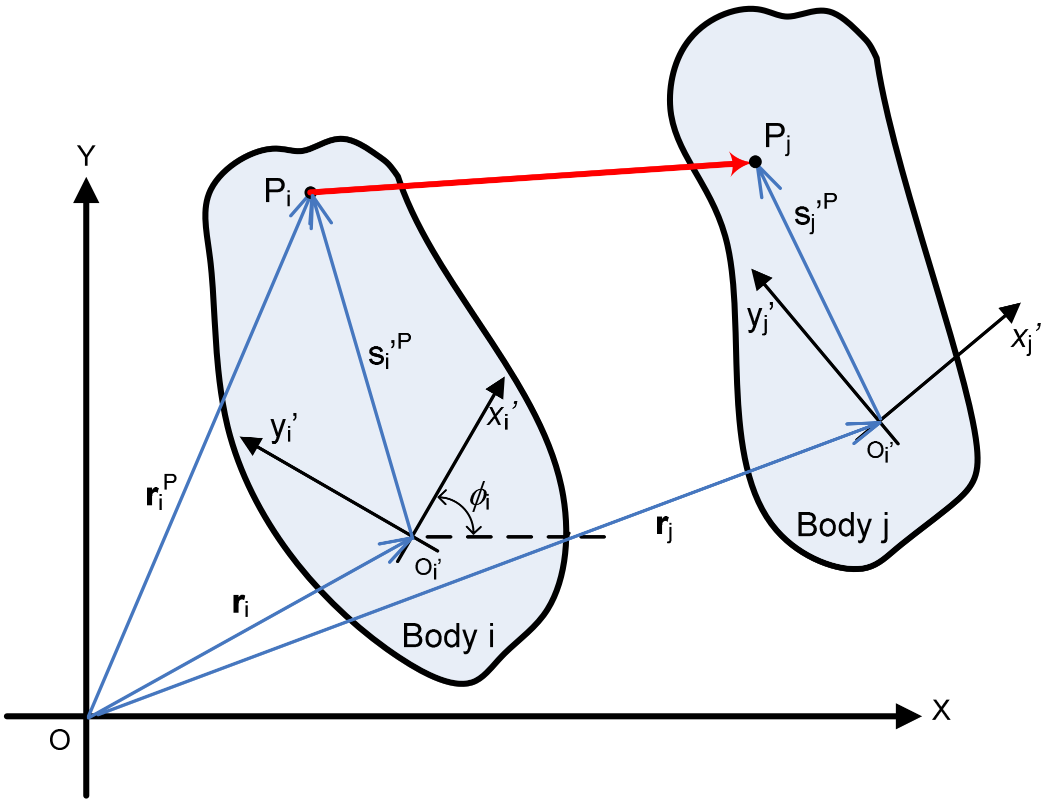
$$\frac{\partial A(\phi)}{\partial \phi} \equiv \mathbf{B}(\phi) = \mathbf{A}(\phi) \cdot \mathbf{R} = \mathbf{R} \cdot \mathbf{A}(\phi) \qquad \text{where} \qquad \mathbf{R} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

- Therefore

$$\mathbf{B}_{ij} = \mathbf{R} \cdot \mathbf{A}_{ij} = \mathbf{A}_{ij} \cdot \mathbf{R} = \begin{bmatrix} -\sin(\phi_j - \phi_i) & -\cos(\phi_j - \phi_i) \\ \cos(\phi_j - \phi_i) & -\sin(\phi_j - \phi_i) \end{bmatrix}$$

Intro: Vector between P_i and P_j

- Something that we'll use a lot: the expression of the vector from P_i to P_j in terms of the Cartesian generalized coordinates **q**



$$\mathbf{q} = \begin{bmatrix} x_i \\ y_i \\ \varphi_i \\ x_j \\ y_j \\ \varphi_j \end{bmatrix} = \begin{bmatrix} \mathbf{r}_i \\ \varphi_i \\ \mathbf{r}_j \\ \varphi_j \end{bmatrix} = \begin{bmatrix} \mathbf{q}_i \\ \mathbf{q}_j \end{bmatrix}$$

$$\overrightarrow{P_iP_j} = \mathbf{r}_j + \mathbf{A}_j\mathbf{s}'^{P_j} - \mathbf{r}_i - \mathbf{A}_i\mathbf{s}'^{P_i}$$

$$\frac{d(\overrightarrow{P_iP_j})}{dt} = \dot{\mathbf{r}}_j + \dot{\phi}_j\mathbf{B}_j\mathbf{s}'^{P_j} - \dot{\mathbf{r}}_i - \dot{\phi}_i\mathbf{B}_i\mathbf{s}'^{P_i}$$

$$\frac{d^2(\overrightarrow{P_iP_j})}{dt^2} = \ddot{\mathbf{r}}_j + \ddot{\phi}_j\mathbf{B}_j\mathbf{s}'^{P_j} - \dot{\phi}_j^2\mathbf{A}_j\mathbf{s}'^{P_j} - \ddot{\mathbf{r}}_i - \ddot{\phi}_i\mathbf{B}_i\mathbf{s}'^{P_i} + \dot{\phi}_i^2\mathbf{A}_i$$

Relative x-constraint

- Step 1: In layman's words, the difference between the x coordinates of point P_j and point P_i should stay constant and equal to some known value C_1

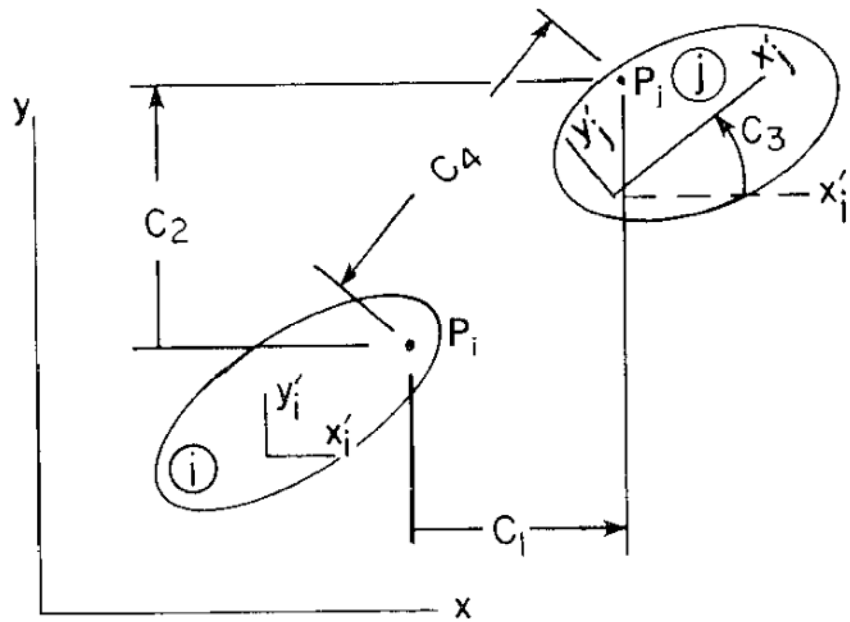


Figure 3.3.1 Simple constraints.

- Step 2: Identify $\Phi^{rx(i,j)}=0$
- Step 3: $\Phi^{rx(i,j)}_{\mathbf{q}} = ?$
- Step 4: $\nu^{rx(i,j)} = ?$
- Step 5 $\gamma^{rx(i,j)} = ?$

Relative y-constraint

- Step 1: The difference between the y coordinates of point P_j and point P_i should stay constant and equal to some known value C_2

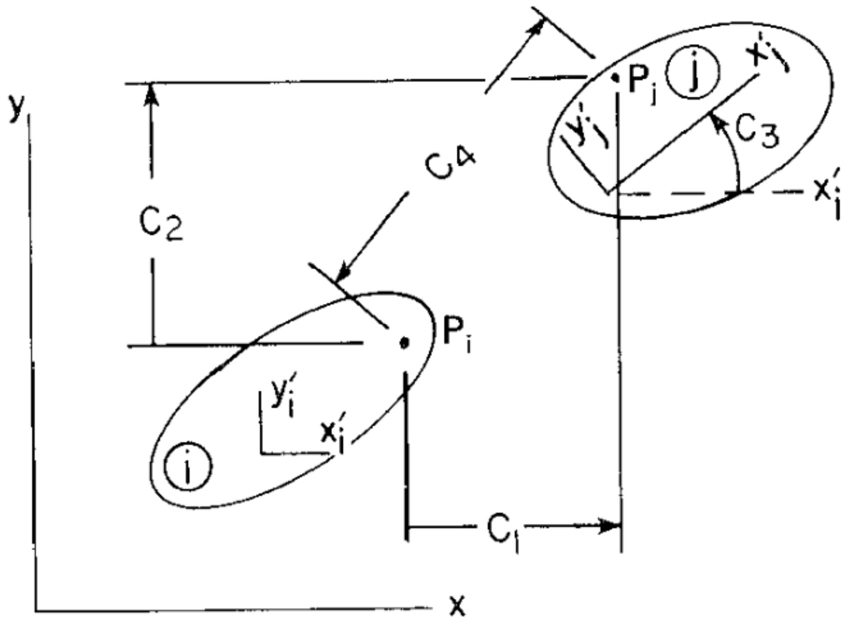


Figure 3.3.1 Simple constraints.

- Step 2: Identify $\Phi^{ry(i,j)}=0$
- Step 3: $\Phi^{ry(i,j)}_{\mathbf{q}} = ?$
- Step 4: $\nu^{ry(i,j)} = ?$
- Step 5: $\gamma^{ry(i,j)} = ?$

Relative angle-constraint

- Step 1: The difference between the orientation angles of the RFs associated with bodies i and j stays constant and equal to some known value C_3

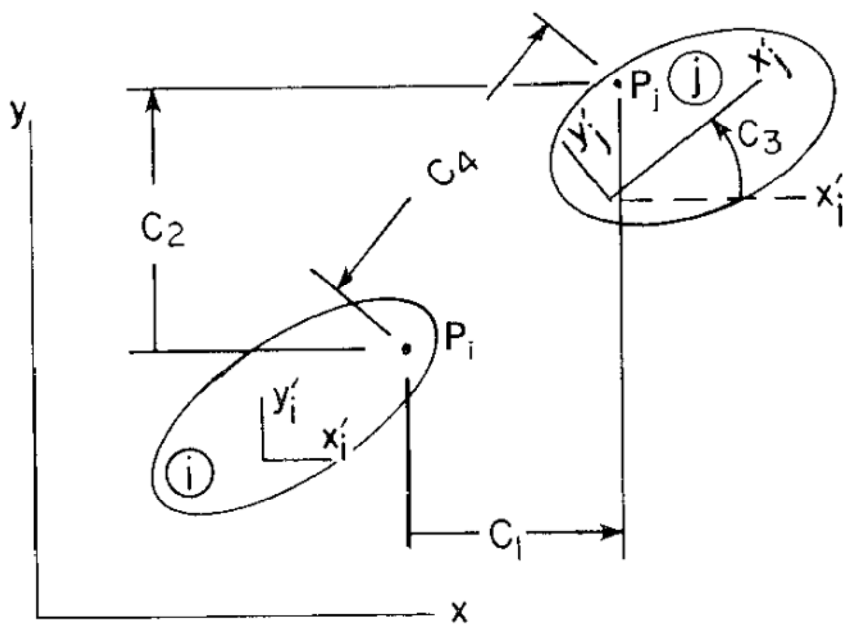


Figure 3.3.1 Simple constraints.

- Step 2: Identify $\Phi^{r\phi(i,j)}=0$
- Step 3: $\Phi^{r\phi(i,j)}_{\mathbf{q}} = ?$
- Step 4: $\nu^{r\phi(i,j)} = ?$
- Step 5: $\gamma^{r\phi(i,j)} = ?$

Relative distance-constraint

- Step 1: The distance between two points P_j and point P_i should stay constant and equal to some known value C_4

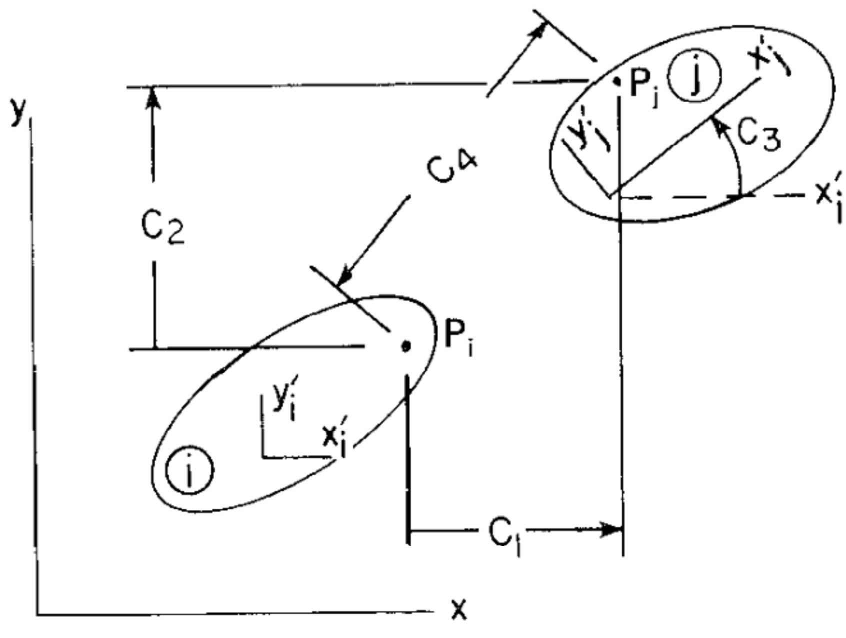


Figure 3.3.1 Simple constraints.

- Step 2: Identify $\Phi^{rd\,(i,j)}=0$
- Step 3: $\Phi^{rd\,(i,j)}_{\mathbf{q}} = ?$
- Step 4: $\nu^{rd\,(i,j)} = ?$
- Step 5: $\gamma^{rd\,(i,j)} = ?$