ME451
Kinematics and Dynamics of Machine Systems

Basic Concepts in Planar Kinematics - 3.1
Absolute Kinematic Constraints – 3.2
Relative Kinematic Constraints – 3.3

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“There is no reason for any individual to have a computer in their home.”
Before we get started…

- **Last time:**
  - Computing the velocity and acceleration of a point attached to a moving rigid body
  - Absolute vs. relative generalized coordinates
  - Start Chapter 3: Kinematics Analysis

- **Today:**
  - Wrap up high level discussion of Kinematics Analysis after introducing the concept of Jacobian
  - Start discussion on how to formulate Kinematic constraints associated with a mechanism
    - Absolute kinematic constraints
    - Relative kinematic constraints

- **Assignment 4 due one week from today:**
  - Problems 2.6.1, 3.1.1, 3.1.2, 3.1.3
  - ADAMS and MATLAB components emailed to you

- **Assignment 3 due today**
  - Problems due in class
  - MATLAB and ADAMS part due at 23:59 PM
Example 3.1.1

- A motion $\phi_1 = 4t^2$ is applied to the pendulum
- Use Cartesian generalized coordinates
- Formulate the velocity analysis problem
- Formulate the acceleration analysis problem
Kinematic Analysis Stages

- **Position Analysis Stage**
  - Challenging

- **Velocity Analysis Stage**
  - Simple

- **Acceleration Analysis Stage**
  - OK

To take care of all these stages, ONE step is critical:
- Write down the constraint equations associated with the joints present in your mechanism
- Once you have the constraints, the rest is boilerplate
Once you have the constraints…

(Going beyond the critical step)

- The three stages of Kinematics Analysis: position analysis, velocity analysis, and acceleration analysis they each follow *very* similar recipes for finding for each body of the mechanism its position, velocity and acceleration, respectively

- ALL STAGES RELY ON THE CONCEPT OF JACOBIAN MATRIX:
  - $\Phi_q$ – the partial derivative of the constraints wrt the generalized coordinates

- ALL STAGES REQUIRE THE SOLUTION OF A SYSTEM OF EQUATIONS

$$\Phi_q \ x = b$$

The Details…

- As we pointed out, it all boils down to this:
  - Step 1: Before anything, write down the constraint equations associated with your model
  - Step 2: For each stage, construct $\Phi_q$ and the specific $b$, then solve for $x$

- So how do you get the position configuration of the mechanism?
  - Kinematic Analysis key observation: The number of constraints (kinematic and driving) should be equal to the number of generalized coordinates
    - This is, NDOF=0, a prerequisite for Kinematic Analysis

$$\Phi(q, t) = \begin{bmatrix} \Phi^K(q) \\ \Phi^D(q, t) \end{bmatrix}_{nc \times 1} = 0$$

$q \in \mathbb{R}^{nc}$

$\Phi : \mathbb{R}^{nc+1} \rightarrow \mathbb{R}^{nc}$

IMPORTANT: This is a nonlinear system with $nc$ equations and $nc$ unknowns that you must solve to find $q$
Getting the Velocity and Acceleration of the Mechanism

- Previous slide taught us how to find the positions $\mathbf{q}$
  - At each time step $t_k$, generalized coordinates $\mathbf{q}_k$ are the solution of a nonlinear system

- Take one time derivative of constraints $\Phi(q,t)$ to obtain the **velocity equation**:

  \[
  \frac{d}{dt} \Phi(q, t) = 0 \quad \Rightarrow \quad \Phi_q \dot{q} = -\Phi_t
  \]

- Take yet one more time derivative to obtain the **acceleration equation**:

  \[
  \ddot{\Phi} = \frac{d^2}{dt^2} \Phi(q, t) = 0 \quad \Rightarrow \quad \Phi_q \ddot{q} = -\left(\Phi_q \dot{q}\right)_q \dot{q} - 2\Phi_{qt} \dot{q} - \Phi_{tt}
  \]

- NOTE: Getting right-hand side of acceleration equation is tedious
Producing RHS of Acceleration Eq.
[In light of previous example]

- RHS was shown to be computed as

\[ \ddot{\Phi} = \frac{d^2}{dt^2} \Phi(q, t) = 0 \implies \Phi_q \ddot{q} = - (\Phi_{qq} \dot{q})_q \dot{q} - 2 \Phi_{qt} \dot{q} - \Phi_{tt} \]

- Note that the RHS contains (is made up of) everything that does *not* depend on the generalized accelerations

- Implication:
  - When doing small examples in class, don’t bother to compute the RHS using expression above
    - This is done only in ADAMS, when you shoot for a uniform approach to all problems
  
  - Simply take two time derivatives of your simple constraints and move everything that does *not* depend on acceleration to the RHS
Focus on Geometric Constraints

- Learn how to write kinematic constraints that specify that the location and/or attitude of a body wrt the global (or absolute) RF is constrained in a certain way
  - Sometimes called **absolute** constraints

- Learn how to write kinematic constraints that couple the relative motion of two bodies
  - Sometimes called **relative** constraints
Step 1: Identify a kinematic constraint (revolute, translational, relative distance, etc., i.e., the physical thing) acting between two components of a mechanism.

Step 2: Formulate the algebraic equations that capture that constraint, $\Phi(q) = 0$. This is called “modeling”.

Step 3: Compute the Jacobian (or the sensitivity matrix) $\Phi_q$.

Step 4: Compute $\nu$, the right side of the velocity equation.

Step 5: Compute $\gamma$, the right side of the acceleration equation (ugly…)

This is what we do almost exclusively in Chapter 3 (about two weeks).
Absolute Constraints

- Called “Absolute” since they express constraint between a body in a system and an absolute (ground) reference frame

- Types of Absolute Constraints
  - Absolute position constraints
  - Absolute orientation constraints
  - Absolute distance constraints
Absolute Constraints (Cntd.)

- Absolute position constraints
  - x-coordinate of $P_i$
    \[ x^{P_i} - C_1 = 0 \]
  - y-coordinate of $P_i$
    \[ y^{P_i} - C_2 = 0 \]

- Absolute orientation constraint
  - Orientation $\phi$ of body
    \[ \phi_i - C_3 = 0 \]

\[
\mathbf{r}^{P_i} = \mathbf{r}_i + \mathbf{A}_i \mathbf{s'}^{P_i} \quad \Rightarrow \quad \mathbf{r}^{P_i} = \begin{bmatrix} x_i \\ y_i \end{bmatrix} + \begin{bmatrix} \cos \phi_i & -\sin \phi_i \\ \sin \phi_i & \cos \phi_i \end{bmatrix} \begin{bmatrix} x_i' \\ y_i' \end{bmatrix}
\]
Absolute x-constraint

- Step 1: the absolute x component of the location of a point $P_i$ in an absolute (or global) reference frame stays constant, and equal to some known value $C_1$

- Step 2: Identify $\Phi^{ax(i)} = 0$

- Step 3: $\Phi^{ax(i)}_q = ?$

- Step 4: $v^{ax(i)} = ?$

- Step 5: $\gamma^{ax(i)} = ?$

Figure 3.2.2 Constraints on absolute coordinates of point $P_i$ and on angular orientation.

NOTE: The same approach is used to get the y- and angle-constraints
Absolute distance-constraint

- Step 1: the distance from a point $P_i$ to an absolute (or global) reference frame stays constant, and equal to some known value $C_4$

- Step 2: Identify $\Phi^{d_x(i)} = 0$

- Step 3: $\Phi^{d_x(i)}_q = ?$

- Step 4: $\nu^{d_x(i)} = ?$

- Step 5: $\gamma^{d_x(i)} = ?$

**Figure 3.2.1** Constraint that distance from point $P$ to $(C_1, C_2)$ equals $C_3$. 