ME451
Kinematics and Dynamics of Machine Systems

Vel. And Acc. of a Fixed Point in Moving Frame - 2.6
Absolute vs. Relative Generalized Coordinates
Basic Concepts in Planar Kinematics - 3.1

September 27, 2011

“I think there is a world market for maybe five computers.”
T. J. Watson, chairman of IBM, 1943.
Before we get started...

- Last time:
  - Computing the partial derivative of functions
  - Discuss chain rule for taking time derivatives
  - Time derivatives of vectors and matrices

- Today:
  - Computing the velocity and acceleration of a point attached to a moving rigid body
  - Absolute vs. relative generalized coordinates
  - Start Chapter 3: Kinematics Analysis

- MATLAB Assignment
  - Lays the foundation for the model definition parser
    - This will be part of your simEngine2D used to read in a model of a mechanism (mechanical system)

- Assignment due on Th
  - Problems due in class
  - MATLAB and ADAMS part due at 23:59 PM on Th
  - Use forum for questions
The problem at hand:

- Rigid body, colored in blue
- This body moves in space, and the location of a point P attached to this body is a function of time and is represented by \( \mathbf{r}_P(t) \)

Question: What is the velocity of P?

- Equivalently, what is the time derivative of \( \mathbf{r}_P(t) \)
Velocity of a Fixed Point in a Moving Frame

- Something to keep in mind: we’ll manipulate quantities that depend on the generalized coordinates, which in turn depend on time. Specifically,
  - Orientation matrix $A$ depends on the generalized coordinate $\phi$, which depends on time $t$
    - This is because the body experiences a rotational motion
  - Vector $r = [x(t) \ y(t)]^T$ that locates the LRF in the GRF depends on time $t$
    - This is because the body experiences also a translational motion

- We’ll use next the time and partial derivatives of previous lecture

\[ r^P = r + A(\phi) \cdot s'^P \]

\[ \downarrow \text{ (Take time derivative)} \]

\[ \dot{r}^P = \dot{r} + \dot{A}(\phi) \cdot s'^P = \dot{r} + \dot{\phi}B(\phi) \cdot s'^P \]
Matrices of Interest

- **Skew Symmetric Matrix $R$:**

  $$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

  Note that when applied to a vector, this rotation matrix produces a new vector that is perpendicular to the original vector (counterclockwise rotation)

  $$\forall \mathbf{v}, \quad \mathbf{v}^\perp = R\mathbf{v}$$

- **The $B$ matrix:**

  $$B(\phi) = \begin{bmatrix} -\sin \phi & -\cos \phi \\ \cos \phi & -\sin \phi \end{bmatrix}$$

  The $B$ matrix is always defined in conjunction with an $A$ matrix (or a reference frame, for that matter). It is defined as

  $$B(\phi) \equiv \frac{\partial A(\phi)}{\partial \phi}$$
Quick Remarks

- Note that the following identities hold:

$$\mathbf{R}^2 = \mathbf{R} \times \mathbf{R} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\mathbf{I}_2$$

$$\mathbf{B}(\phi) = \mathbf{A}(\phi) \cdot \mathbf{R} = \mathbf{R} \cdot \mathbf{A}(\phi)$$

$$\dot{\mathbf{A}}(\phi) = \dot{\phi} \mathbf{B}(\phi)$$

$$\dot{\mathbf{B}}(\phi) = -\dot{\phi} \mathbf{A}(\phi)$$

- As a matter of convenience and to make the notation more terse, we typically don’t show the dependency of \( \mathbf{A} \) and \( \mathbf{B} \) on \( \phi \)
Acceleration of a Fixed Point in a Moving Frame

- Same idea as for velocity, except that you need two time derivatives to get accelerations

$$\ddot{r}^P = \ddot{r} + \ddot{\phi}B_s^P - \dot{\phi}^2 A_s^P$$
Example

- You are given:
  - Position of point P in LRF:
  - Position of LRF
  - Orientation of LRF
  - Translational velocity of LRF
  - Angular velocity of LRF
  - Translational acceleration of LRF
  - Angular acceleration of LRF

LRF – local reference frame
GRF – global reference frame

- You are asked:
  - The position of P in GRF
  - The velocity of P in GRF
  - The acceleration of P in GRF

\[ \mathbf{s'}^P = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \]

\[ \mathbf{r} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \phi = \frac{\pi}{3} \]

\[ \mathbf{r}' = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \quad \dot{\phi} \equiv \omega = 3 \]

\[ \ddot{\mathbf{r}} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad \ddot{\phi} = \dot{\omega} = 5 \]
Absolute (Cartesian) Generalized Coordinates

vs.

Relative Generalized Coordinates
Generalized Coordinates

Generalized coordinates: What are they?
- A set of quantities (variables) that allow you to uniquely determine the state of the mechanism
  - You need to know the location of each body
  - You need to know the orientation of each body

- The quantities (variables) are bound to change in time since our mechanism moves
  - In other words, the generalized coordinates are functions of time

- The rate at each the generalized coordinates change is capture by the set of generalized velocities
  - Most often, obtained as the straight time derivative of the generalized coordinates
  - Sometimes this is not the case though
    - Example: in 3D Kinematics, there is generalized coordinate whose time derivative is the angular velocity

Important remark: there are multiple ways of choose the set of generalized coordinates that describe the state of your mechanism
- We’ll briefly look at two choices next
Example (RGC)

- Use the array $q$ of generalized coordinates to locate the point B in the GRF (Global Reference Frame)
- Write down the constraint on the motion of point B

$$q = \begin{bmatrix} \theta_1 \\ \theta_{12} \end{bmatrix}$$
Example (AGC)

- Use array $q$ of generalized coordinates to locate the point B in the GRF (Global Reference Frame)
- Write down the constraint on the motion of point B

\[
q = \begin{bmatrix}
x_2 \\
y_2 \\
\theta_2
\end{bmatrix}
\]

\[\theta_1, \theta_2\]
Relative vs. Absolute Generalized Coordinates

- A consequential question:
  - Where was it easier to come up with position of point B?

- First Approach (Example RGC) – relies on relative coordinates:
  - Angle $\theta_1$ uniquely specified both position and orientation of body 1
  - Angle $\theta_{12}$ uniquely specified the position and orientation of body 2 with respect to body 1
    - To locate B wrt global RF, first I position it with respect to body 1 (drawing on $\theta_{12}$), and then locate the latter wrt global RF (based on $\theta_1$)
    - Note that if there were 100 bodies, I would have to position wrt to body 99, which then I locate wrt body 98, …, and finally position wrt global RF
Relative vs. Absolute Generalized Coordinates (Cntd)

- Second Approach (Example AGC) – relies on absolute (and Cartesian) generalized coordinates:
  - $x_1, y_1, \theta_1$ position and orient body 1 wrt GRF (global RF)
  - $x_2, y_2, \theta_2$ position and orient body 2 wrt GRF (global RF)
    - To express the location of B is then very straightforward, use only $x_2, y_2, \theta_2$ and local information (local position of B in body 2)

- For AGC, you handle many generalized coordinates
  - 3 for each body in the system (six for this example)
Relative vs. Absolute Generalized Coordinates (Cntd.)

- Conclusion for AGC and RGC:
  - There is no free lunch:
    - AGC: easy to express locations but many GCs
    - RGC: few GCs but cumbersome process of locating B
  - We’ll use AGC, the math is simple, let the computer keep track of the multitude of GCs…
- RGC common in robotics and molecular dynamics
- AGC common in multibody dynamics
End Chapter 2
Begin Chapter 3
(Kinematics)
What is Kinematics?

- Study of the position, velocity, and acceleration of a system of interconnected bodies that make up a mechanism, independent of the forces that produce the motion
Why is Kinematics Important?

- It can be an end in itself…
  - *Kinematic Analysis* - Interested how components of a certain mechanism move when motion[s] are applied
  - *Kinematic Synthesis* – Interested in finding how to design a mechanism to perform a certain operation in a certain way
    - NOTE: we only focus on Kinematic Analysis

- It is also an essential ingredient when formulating the Kinetic problem (so called Dynamics Analysis, discussed in Chapter 6)

- In general, people are more interested in the Dynamic Analysis rather than in the Kinematic Analysis
Purpose of Rest of the Lecture

- Before getting lost in the details of Kinematics Analysis:
  - Present a collection of terms that will help understand the “language” of Kinematics
  - Give a 30,000 feet perspective of things to come and justifies the need for the material presented over the next 2-3 weeks

- Among the concepts introduced today, here are the more important ones:
  - Constraint equations (as a means to defining the geometry associated with the motion of a mechanism)
  - Jacobian matrix (or simply, the Jacobian)
Nomenclature

- Rigid body

- Body-fixed Reference Frame (also called Local Reference Frame, LRF)

- Generalized coordinates

\[
q = \begin{bmatrix}
q_1 \\
q_2 \\
\vdots \\
q_{nc}
\end{bmatrix} \in \mathbb{R}^{nc}
\]

- Cartesian generalized coordinates

\[
q_i = \begin{bmatrix}
x \\
y \\
\phi
\end{bmatrix}_i
\]

- NOTE: for a mechanism with nb bodies, the number of Cartesian generalized coordinates associated is

\[
nc = 3 \cdot nb
\]
Constraints

- What are they, and what role do they play?
  - A collection of equations that if satisfied, they coerce the bodies in the model to move like the bodies of the mechanism

- Most important thing in relation to constraints:
  - For each joint in the model, the equations of constraint that you use must imply the relative motion allowed by the joint
  - Keep in mind: the way you model should resemble the physical system

- Taxonomy of constraints:
  - Holonomic vs. Nonholonomic constraints
  - Scleronomic vs. Rheonomic constraints
    - Sometimes called Kinematic vs Driving constraints
Constraints

[Cntd.]

- Holonomic vs. Nonholonomic
  - Holonomic constraints are constraints that only involve generalized coordinates (almost all of the constraints fall in this category – this is what we use in 451)
  - Nonholonomic constraints – constraints that also involve the time derivative of generalized coordinates (generalized velocities). Example: roll without slip motion

- Scleronomic (Kinematic) vs. Rheonomic (Driving)
  - Scleronomic (Kinematic) – constraints that do not depend *explicitly* on time but rather exclusively on generalized coordinates and possibly their time derivative
    
    Notation Used: \( \Phi^K(q) \)
  
  - Rheonomic (Driving) – constraints that depend explicitly on time. They actually define a motion
    
    Notation Used: \( \Phi^D(q, t) \)
Degrees of Freedom

- The number of degrees of freedom of a mechanism is qualitatively related to the difference between the number of generalized coordinates and the number of constraints that these coordinates must satisfy.

  - Kinematic Degrees of Freedom (KDOF): the difference between the number of generalized coordinates and the number of Kinematic (Scleronomic) constraints.
    - It is an attribute of the model, and it is independent of generalized coordinates used to represent the time evolution of the mechanism.

  - Net Degrees of Freedom (NDOF): the difference between the number of generalized coordinates and the total number of constraints, be them Kinematic (Scleronomic) or Driving (Rheonomic).
    - Depends on how many motions you decide to specify for the parts of the mechanism.

- **IMPORTANT OBSERVATION:** For carrying out Kinematic Analysis, a number of KDOF motions should be specified so that in the end we have NDOF=0.
Motion: Causes

- How can one set a mechanical system in motion?

  - For a system with $ndof$ degrees of freedom, specify NDOF additional driving constraints (one per degree of freedom) that uniquely determine $q(t)$ as the solution of an algebraic problem (Kinematic Analysis).

  - Specify/Apply a set of forces acting upon the mechanism, in which case $q(t)$ is found as the solution of a differential problem (Dynamic Analysis).

 Ignore this for now…
Example 3.1.1

- A pin (revolute) joint present at point O
- A motion \( \phi_1 = 4t^2 \) is applied to the pendulum
- Specify the set of constraints associated with this model
  - Use Cartesian coordinates
- Write down the Kinematic and Driving constraints
  - Specify the value of KDOF and NDOF