ME451
Kinematics and Dynamics of Machine Systems

Vel. And Acc. of a Fixed Point in Moving Frame - 2.6

Absolute vs. Relative Generalized Coordinates

Basic Concepts in Planar Kinematics - 3.1

September 27, 2011

"I think there is a world market for maybe five computers."
T. J. Watson, chairman of IBM, 1943.

Before we get started…

- Last time:
  - Computing the partial derivate of functions
  - Discuss chain rule for taking time derivatives
  - Time derivatives of vectors and matrices

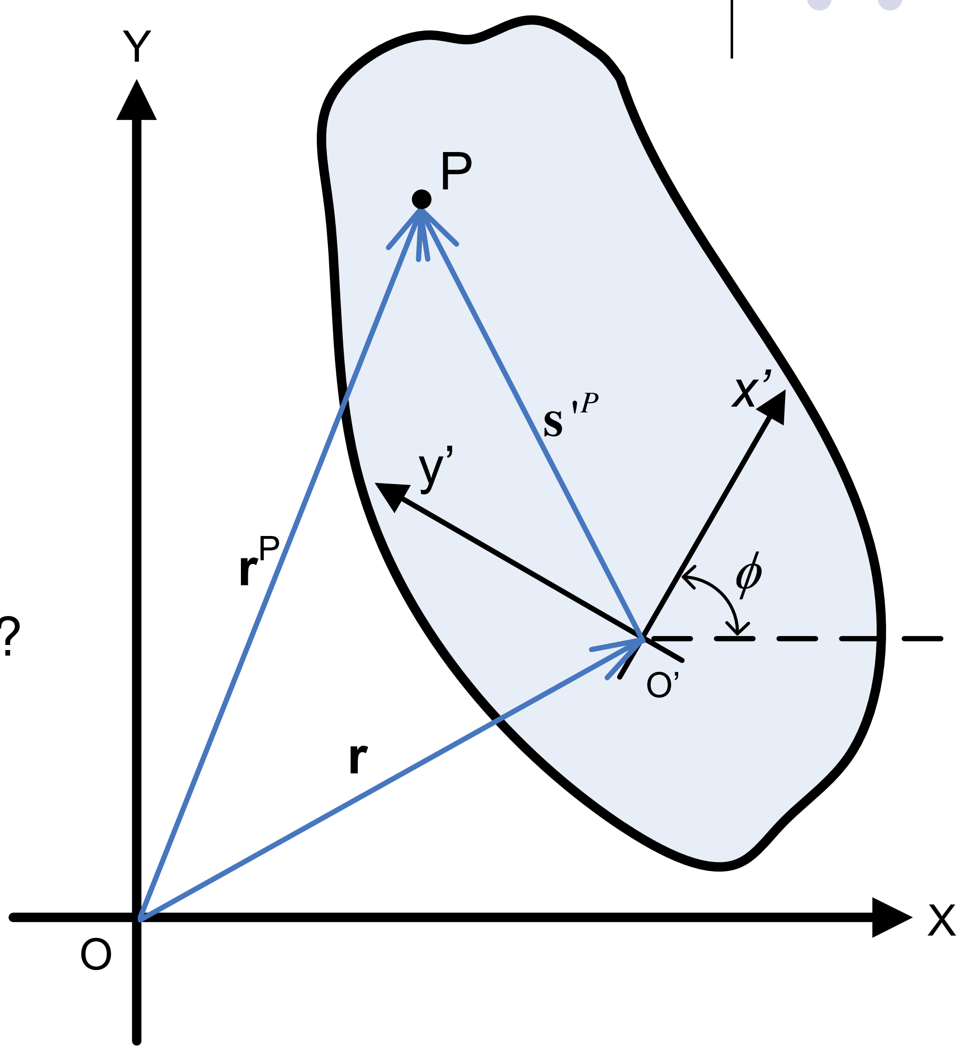
- Today:
  - Computing the velocity and acceleration of a point attached to a moving rigid body
  - Absolute vs. relative generalized coordinates
  - Start Chapter 3: Kinematics Analysis

- MATLAB Assignment
  - Lays the foundation for the model definition parser
    - This will the part of your simEngine2D used to read in a model of a mechanism (mechanical system)

- Assignment due on Th
  - Problems due in class
  - MATLAB and ADAMS part due at 23:59 PM on Th
  - Use forum for questions

Velocity of a Point Fixed to a Moving Frame (Chapter 2.6)

- The problem at hand:
  - Rigid body, colored in blue
  - This body moves in space, and the location of a point P attached to this body is a function of time and is represented by $\mathbf{r}^P(t)$
- Question: What is the velocity of P ?
  - Equivalently, what is the time derivative of $\mathbf{r}^P(t)$



<u>Velocity</u> of a Fixed Point in a Moving Frame

- Something to keep in mind: we'll manipulate quantities that depend on the generalized coordinates, which in turn depend on time. Specifically,
  - Orientation matrix **A** depends on the generalized coordinate ϕ, which depends on time t
    - This is because body experiences a rotational motion
  - Vector **r** =[x(t) y(t)]^T that locates the LRF in the GRF depends on time t
    - This is because the body experiences also a translational motion

- We'll use next the <u>time</u> and <u>partial</u> derivates of previous lecture

$$\mathbf{r}^P = \mathbf{r} + \mathbf{A}(\phi) \cdot \mathbf{s}'^P$$

$$\Downarrow \qquad \text{(Take time derivative)}$$

$$\dot{\mathbf{r}}^P = \dot{\mathbf{r}} + \dot{\mathbf{A}}(\phi) \cdot \mathbf{s}'^P = \dot{\mathbf{r}} + \dot{\phi}\mathbf{B}(\phi) \cdot \mathbf{s}'^P$$

Matrices of Interest

- Skew Symmetric Matrix **R**:

$$\mathbf{R} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

  - Note that when applied to a vector, this rotation matrix produces a new vector that is perpendicular to the original vector (counterclockwise rotation)

$$\forall \mathbf{v}, \quad \mathbf{v}^{\perp} = \mathbf{R}\mathbf{v}$$

- The **B** matrix:

$$\mathbf{B}(\phi) = \begin{bmatrix} -\sin\phi & -\cos\phi \\ \cos\phi & -\sin\phi \end{bmatrix}$$

  - The B matrix is always defined in conjunction with an **A** matrix (or a reference frame, for that matter). It is defined as

$$\mathbf{B}(\phi) \equiv \frac{\partial \mathbf{A}(\phi)}{\partial \phi}$$

Quick Remarks

- Note that the following identities hold:

$$\mathbf{R}^2 = \mathbf{R} \times \mathbf{R} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\mathbf{I}_2$$

$$\mathbf{B}(\phi) = \mathbf{A}(\phi) \cdot \mathbf{R} = \mathbf{R} \cdot \mathbf{A}(\phi)$$

$$\dot{\mathbf{A}}(\phi) = \dot{\phi}\mathbf{B}(\phi)$$

$$\dot{\mathbf{B}}(\phi) = -\dot{\phi}\mathbf{A}(\phi)$$

- As a matter of convenience and to make the notation more terse, we typically don't show the dependency of **A** and **B** on ϕ

<u>Acceleration</u> of a Fixed Point in a Moving Frame

- Same idea as for velocity, except that you need two time derivates to get accelerations

$$\dot{\mathbf{r}}^P = \dot{\mathbf{r}} + \dot{\mathbf{A}}\mathbf{s}'^P = \dot{\mathbf{r}} + \dot{\phi}\mathbf{B}\mathbf{s}'^P$$

$$\Downarrow \quad \text{(Take time derivative)}$$

$$\ddot{\mathbf{r}}^P = \ddot{\mathbf{r}} + \ddot{\phi}\mathbf{B}\mathbf{s}'^P - \dot{\phi}^2\mathbf{A}\mathbf{s}'^P$$

Example

- You are given:
  - Position of point P in LRF:
  - Position of LRF
  - Orientation of LRF
  - Translational velocity of LRF
  - Angular velocity of LRF
  - Translational acceleration of LRF
  - Angular acceleration of LRF

$$\mathbf{s}'^P = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

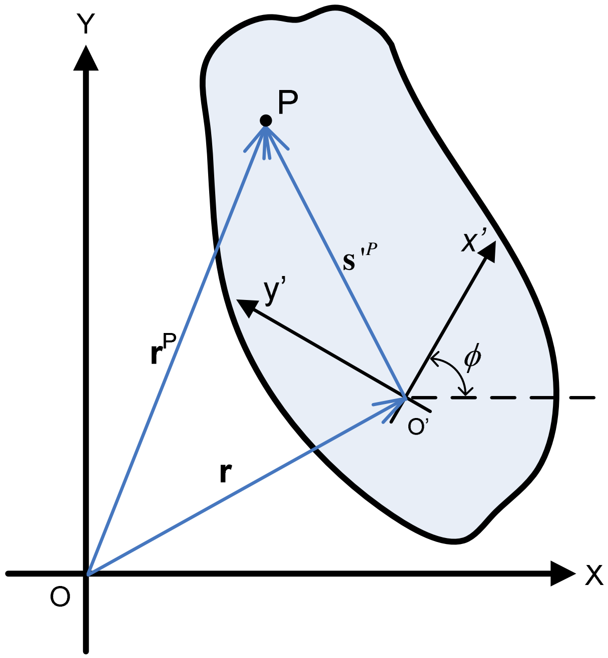
$$\mathbf{r} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad \phi = \frac{\pi}{3}$$

$$\dot{\mathbf{r}} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \qquad \dot{\phi} \equiv \omega = 3$$

$$\ddot{\mathbf{r}} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \qquad \ddot{\phi} = \dot{\omega} = 5$$

- LRF – local reference frame
- GRF – global reference frame

- You are asked:
  - The position of P in GRF
  - The velocity of P in GRF
  - The acceleration of P in GRF



$$\mathbf{r}^P = ?$$

$$\dot{\mathbf{r}}^P = ?$$

$$\ddot{\mathbf{r}}^P = ?$$

Absolute (Cartesian) Generalized Coordinates

vs.

Relative Generalized Coordinates

Generalized Coordinates

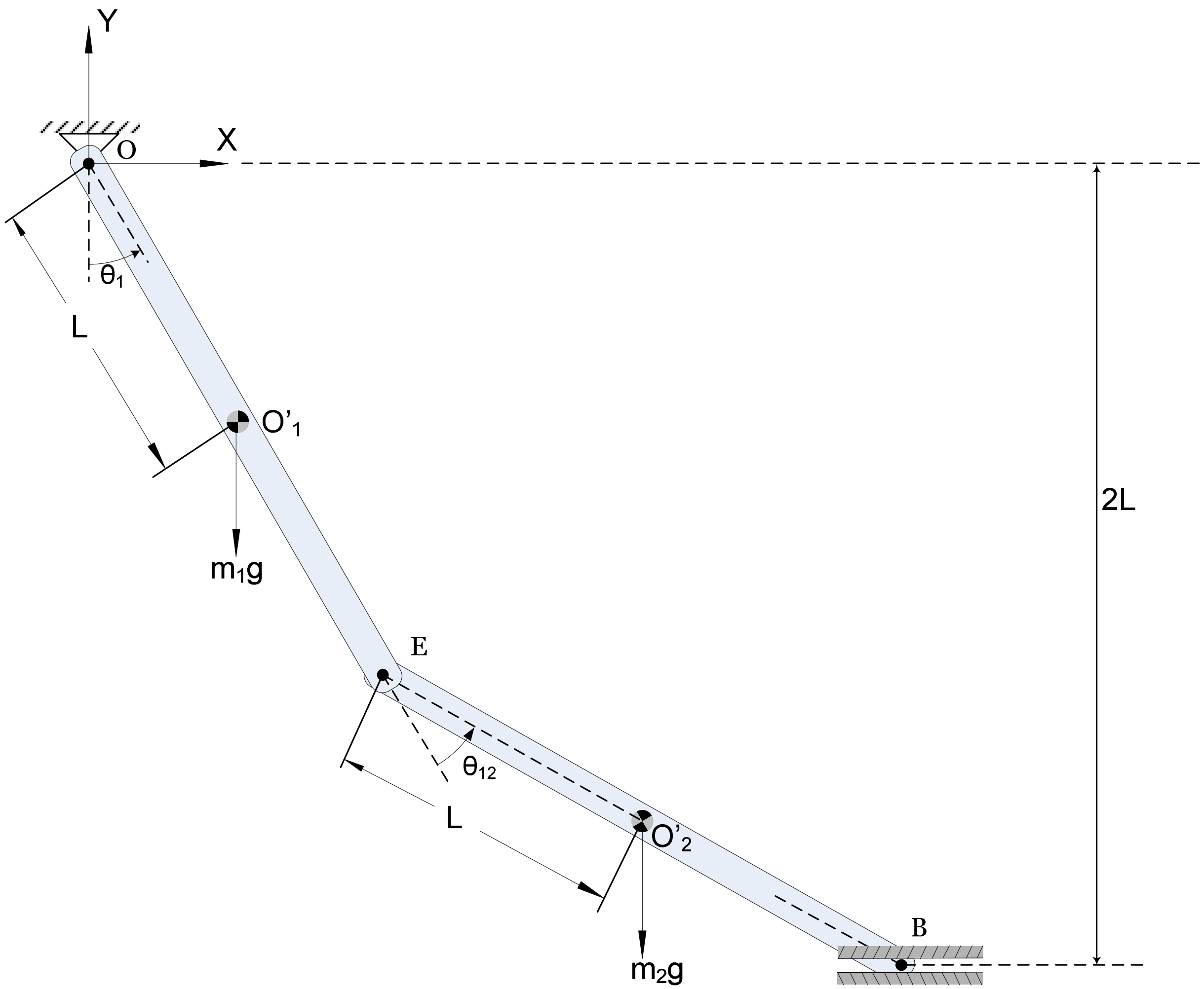
[General Comments]

- Generalized coordinates: What are they?
    - A set of quantities (variables) that allow you to uniquely determine the state of the mechanism
        - You need to know the location of each body
        - You need to know the orientation of each body
    - The quantities (variables) are bound to change in time since our mechanism moves
        - In other words, the generalized coordinates are functions of time
    - The rate at each the generalized coordinates change is capture by the set of generalized velocities
        - Most often, obtained as the straight time derivative of the generalized coordinates
        - Sometimes this is not the case though
            - Example: in 3D Kinematics, there is generalized coordinate whose time derivative is the angular velocity
- Important remark: there are multiple ways of choose the set of generalized coordinates that describe the state of your mechanism
    - We'll briefly look at two choices next

[PO]

Example (RGC)

- Use the array **q** of generalized coordinates to locate the point B in the GRF (Global Reference Frame)
- Write down the constraint on the motion of point B

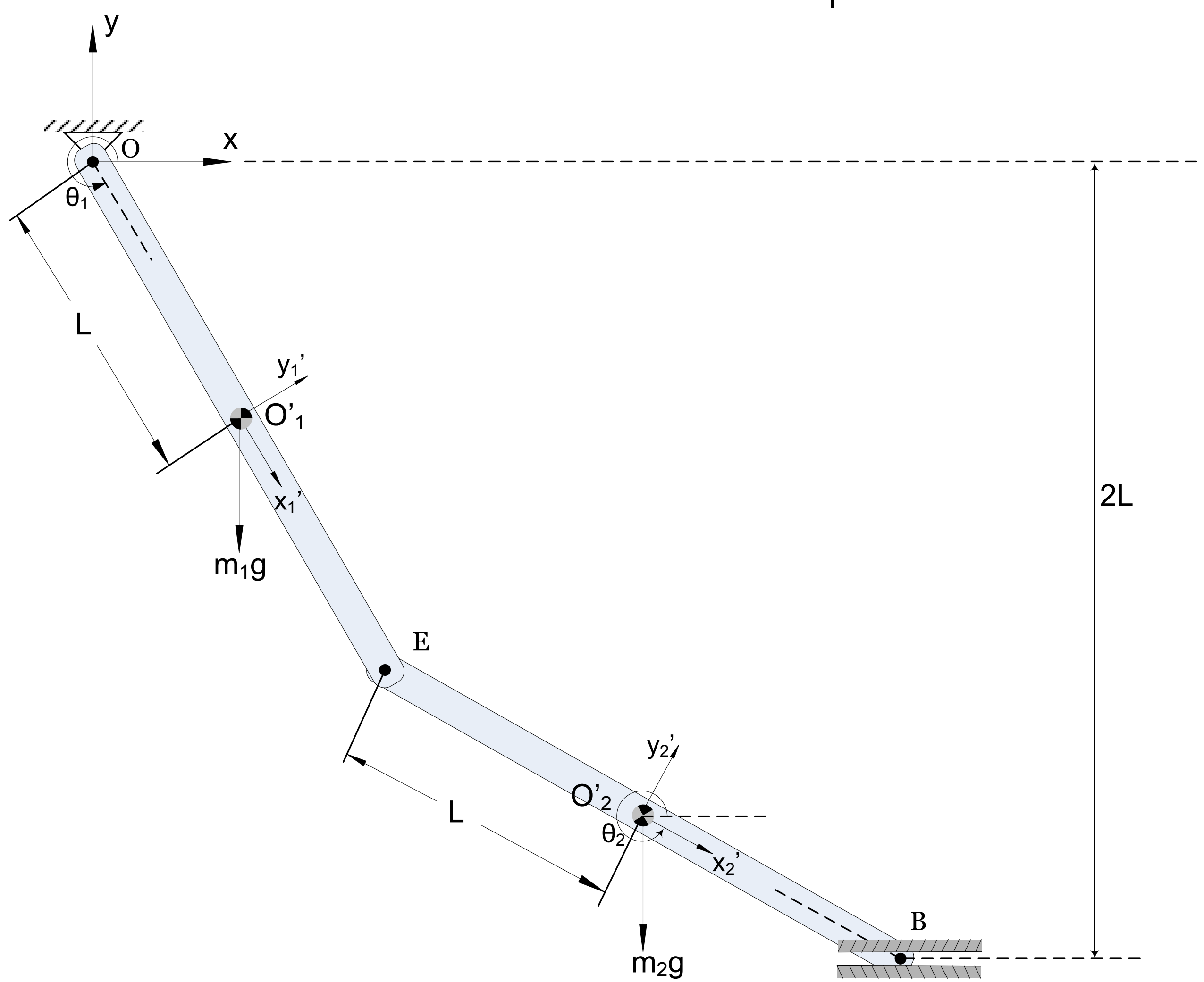


$$\mathbf{q} = \begin{bmatrix} \theta_1 \\ \theta_{12} \end{bmatrix}$$

[PO]

Example (AGC)

- Use array **q** of generalized coordinates to locate the point B in the GRF (Global Reference Frame)
- Write down the constraint on the motion of point B



$$\mathbf{q} = \begin{bmatrix} x_2 \\ y_2 \\ \theta_2 \end{bmatrix}$$

Relative vs. Absolute Generalized Coordinates

- A consequential question:
  - Where was it easier to come up with position of point B?

- First Approach (Example RGC) – relies on relative coordinates:
  - Angle θ_1 uniquely specified both position and orientation of body 1
  - Angle θ_{12} uniquely specified the position and orientation of body 2 with respect to body 1
    - To locate B wrt global RF, first I position it with respect to body 1 (drawing on θ_{12}), and then locate the latter wrt global RF (based on θ_1)
    - Note that if there were 100 bodies, I would have to position wrt to body 99, which then I locate wrt body 98, …, and finally position wrt global RF

Relative vs. Absolute Generalized Coordinates (Cntd)

- Second Approach (Example AGC) – relies on absolute (and Cartesian) generalized coordinates:

  - x_1, y_1, θ_1 position and orient body 1 wrt GRF (global RF)

  - x_2, y_2, θ_2 position and orient body 2 wrt GRF (global RF)
    - To express the location of B is then very straightforward, use only x_2, y_2, θ_2 and local information (local position of B in body 2)

  - For AGC, you handle many generalized coordinates
    - 3 for each body in the system (six for this example)

Relative vs. Absolute Generalized Coordinates (Cntd.)

- Conclusion for AGC and RGC:
  - There is no free lunch:
    - AGC: easy to express locations but many GCs
    - RGC: few GCs but cumbersome process of locating B
  - We'll use AGC, the math is simple, let the computer keep track of the multitude of GCs…
- RGC common in robotics and molecular dynamics
- AGC common in multibody dynamics

End Chapter 2
Begin Chapter 3
(Kinematics)

What is Kinematics?

- Study of the position, velocity, and acceleration of a system of interconnected bodies that make up a mechanism, independent of the forces that produce the motion

Why is Kinematics Important?

- It can be an end in itself…
  - *Kinematic Analysis* - Interested how components of a certain mechanism move when motion[s] are applied
  - *Kinematic Synthesis* – Interested in finding how to design a mechanism to perform a certain operation in a certain way
    - NOTE: we only focus on Kinematic Analysis

- It is also an essential ingredient when formulating the Kinetic problem (so called Dynamics Analysis, discussed in Chapter 6)

- In general, people are more interested in the Dynamic Analysis rather than in the Kinematic Analysis

Purpose of Rest of the Lecture

- Before getting lost in the details of Kinematics Analysis:
  - Present a collection of terms that will help understand the "language" of Kinematics
  - Give a 30,000 feet perspective of things to come and justifies the need for the material presented over the next 2-3 weeks

- Among the concepts introduced today, here are the more important ones:
  - Constraint equations (as a means to defining the geometry associated with the motion of a mechanism)
  - Jacobian matrix (or simply, the Jacobian)

Nomenclature

- Rigid body
- Body-fixed Reference Frame (also called Local Reference Frame, LRF)
- Generalized coordinates $\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ \cdots \\ q_{nc} \end{bmatrix} \in \mathbb{R}^{nc}$
- Cartesian generalized coordinates $\mathbf{q}_i = \begin{bmatrix} x \\ y \\ \phi \end{bmatrix}_i$
  - NOTE: for a mechanism with nb bodies, the number of Cartesian generalized coordinates associated is

$$nc = 3 \cdot nb$$

Constraints

- What are they, and what role do they play?
  - A collection of equations that if satisfied, they coerce the bodies in the model to move like the bodies of the mechanism
- Most important thing in relation to constraints:
  - For each joint in the model, the equations of constraint that you use must imply the relative motion allowed by the joint
  - Keep in mind: the way you **model** should resemble the **physical system**
- Taxonomy of constraints:
  - Holonomic vs. Nonholonomic constraints
  - Scleronomic vs. Rheonomic constraints
    - Sometimes called Kinematic vs Driving constraints

Constraints
[Cntd.]

- Holonomic vs. Nonholonomic
  - Holonomic constraints are constraints that only involve generalized coordinates (almost all of the constraints fall in this category – this is what we use in 451)
  - Nonholonomic constraints – constraints that also involve the time derivative of generalized coordinates (generalized velocities). Example: roll without slip motion

- Scleronomic (Kinematic) vs. Rheonomic (Driving)
  - Scleronomic (Kinematic) – constraints that do not depend *explicitly* on time but rather exclusively on generalized coordinates and possibly their time derivative

    Notation Used: $\mathbf{\Phi}^K(\mathbf{q})$

  - Rheonomic (Driving) – constraints that depend explicitly on time. They actually define a motion

    Notation Used: $\mathbf{\Phi}^D(\mathbf{q}, t)$

Degrees of Freedom

- The number of degrees of freedom of a mechanism is qualitatively related to the difference between the number of generalized coordinates and the number of constraints that these coordinates must satisfy
  - Kinematic Degrees of Freedom (KDOF): the difference between the number of generalized coordinates and the number of Kinematic (Scleronomic) constraints
    - It is an attribute of the model, and it is <u>independent</u> of generalized coordinates used to represent the time evolution of the mechanism
  - Net Degrees of Freedom (NDOF): the difference between the number of generalized coordinates and the total number of constraints, be them Kinematic (Scleronomic) or Driving (Rheonomic)
    - Depends on how many motions you decide to specify for the parts of the mechanism

- <u>IMPORTANT OBSERVATION</u>: For carrying out Kinematic Analysis, a number of KDOF motions should be specified so that in the end we have NDOF=0

Motion: Causes

- How can one set a mechanical system in motion?

  - For a system with *ndof* degrees of freedom, specify NDOF additional driving constraints (one per degree of freedom) that uniquely determine **q**(t) as the solution of an algebraic problem (Kinematic Analysis)

  - Specify/Apply a set of forces acting upon the mechanism, in which case **q**(t) is found as the solution of a differential problem (Dynamic Analysis)

Ignore this for now…

Example 3.1.1

- A pin (revolute) joint present at point O
- A motion $\phi_1 = 4t^2$ is applied to the pendulum
- Specify the set of constraints associated with this model
  - Use Cartesian coordinates
- Write down the Kinematic and Driving constraints
  - Specify the value of KDOF and NDOF