ME451
Kinematics and Dynamics of Machine Systems

Review of Matrix Algebra – 2.2

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Before we get started…

- Due next week:
  - Problems: 2.2.5, 2.2.8. 2.2.10 out of Haug's book (http://sbel.wisc.edu/Courses/ME451/2010/bookHaugPointers.htm)
  - Due on Th:
    - In class, if pen and paper version is submitted
    - At 23:59 PM if electronic form submitted
      - Moving to full electronic submission starting after September
      - See Forum posting for some ideas on how to go about typing equations in your document in Windows

- Last time:
  - Covered Geometric Vectors & operations with them
  - Justified the need for Reference Frames (using a vector basis)
  - Introduced algebraic representation of a vector & related operations
  - Rotation Matrix (for switching from one RF to another RF)

- Today:
  - Dealing with the kinematics of a body: rotation + translation
  - Quick review of matrix/vector algebra
  - Discuss concept of "generalized coordinates"

The Rotation Matrix A

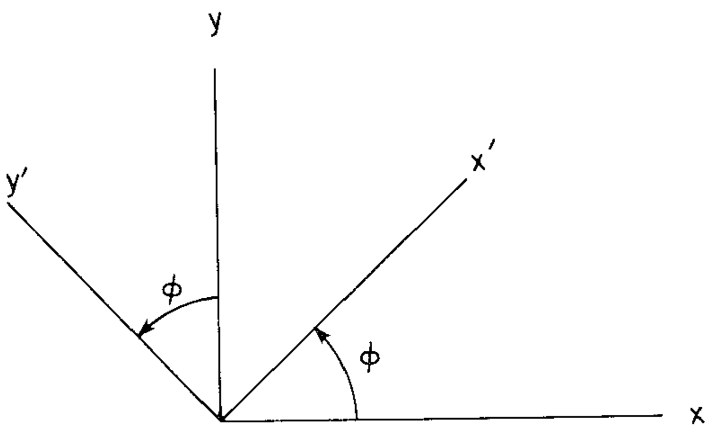


Figure 2.4.1 Two Cartesian reference frames.

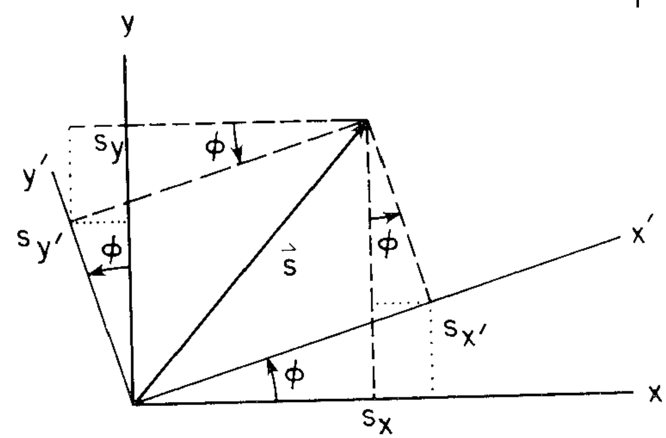


Figure 2.4.2 Vector $\vec{s}$ in two frames.

$$\mathbf{A} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

- Very important observation $\rightarrow$ the matrix **A** is orthonormal:

$$\mathbf{A}^T \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{A}^T = \mathbf{I}_{2\times 2}$$

Important Relation

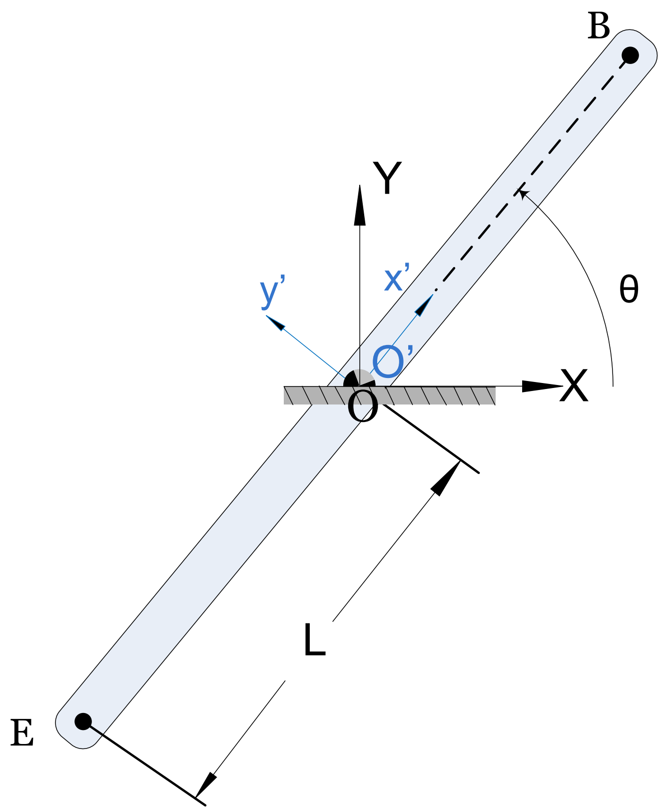
- Expressing a given vector in one reference frame (local) in a different reference frame (global)

$$\mathbf{s} = \mathbf{A}\mathbf{s}'$$

Also called a change of base.

Example 1

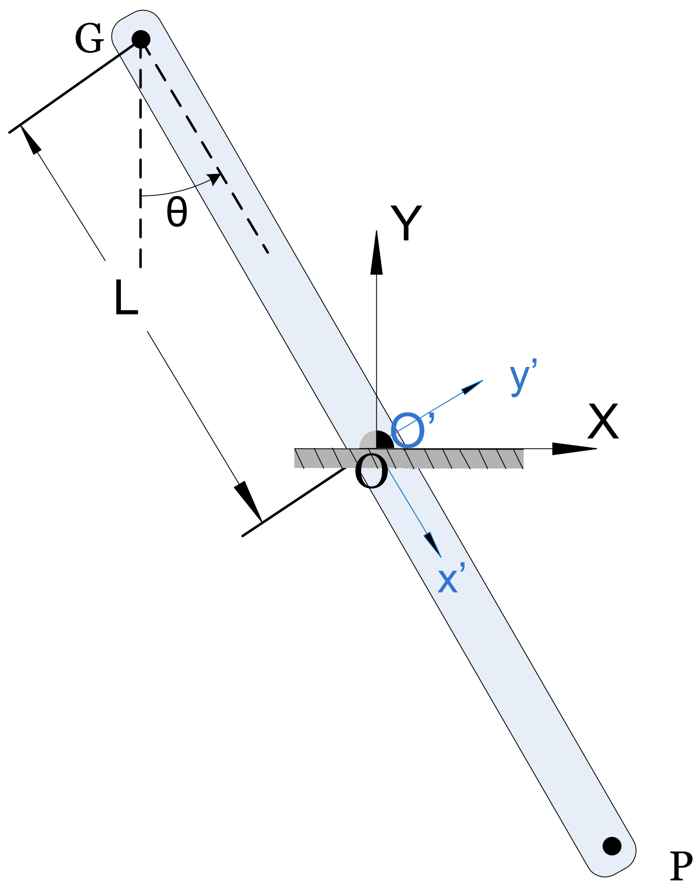
[Deals with the rotation of a body wrt a Global Reference Frame (GRF)]



- Express the geometric vector $\overrightarrow{O'B}$ in the local reference frame O'x'y'.
- Express the same geometric vector in the global reference frame OXY
- Do the same for the geometric vector $\overrightarrow{O'E}$

Example 2

[Deals with the rotation of a body wrt a Global Reference Frame (GRF)]



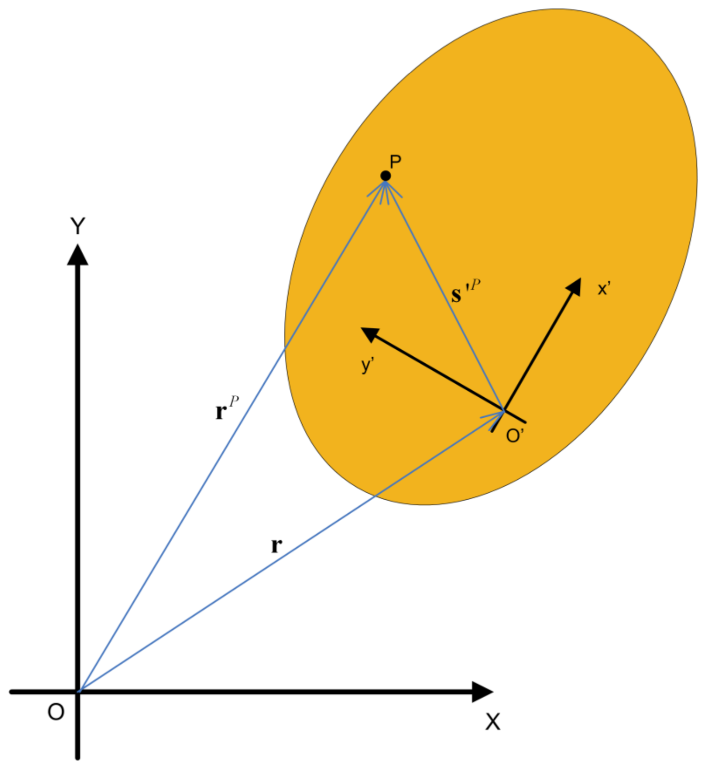
- Express the geometric vector $$\overrightarrow{O'P}$$ in the local reference frame O'x'y'.
- Express the same geometric vector in the global reference frame OXY
- Do the same for the geometric vector $$\overrightarrow{O'G}$$

The Kinematics of a Rigid Body: Handling both Translation + Rotation

- What we just discussed dealt with the case when you are interested in finding the representing the location of a point P when you change the reference frame, but yet the new and old reference frames share the <u>same</u> origin

- What if they don't share the same origin (see picture at right)? How would you represent the position of the point P in this new reference frame?

$$\mathbf{r}^P = \mathbf{r} + \mathbf{s}^P = \mathbf{r} + \mathbf{A}\mathbf{s}'^P$$

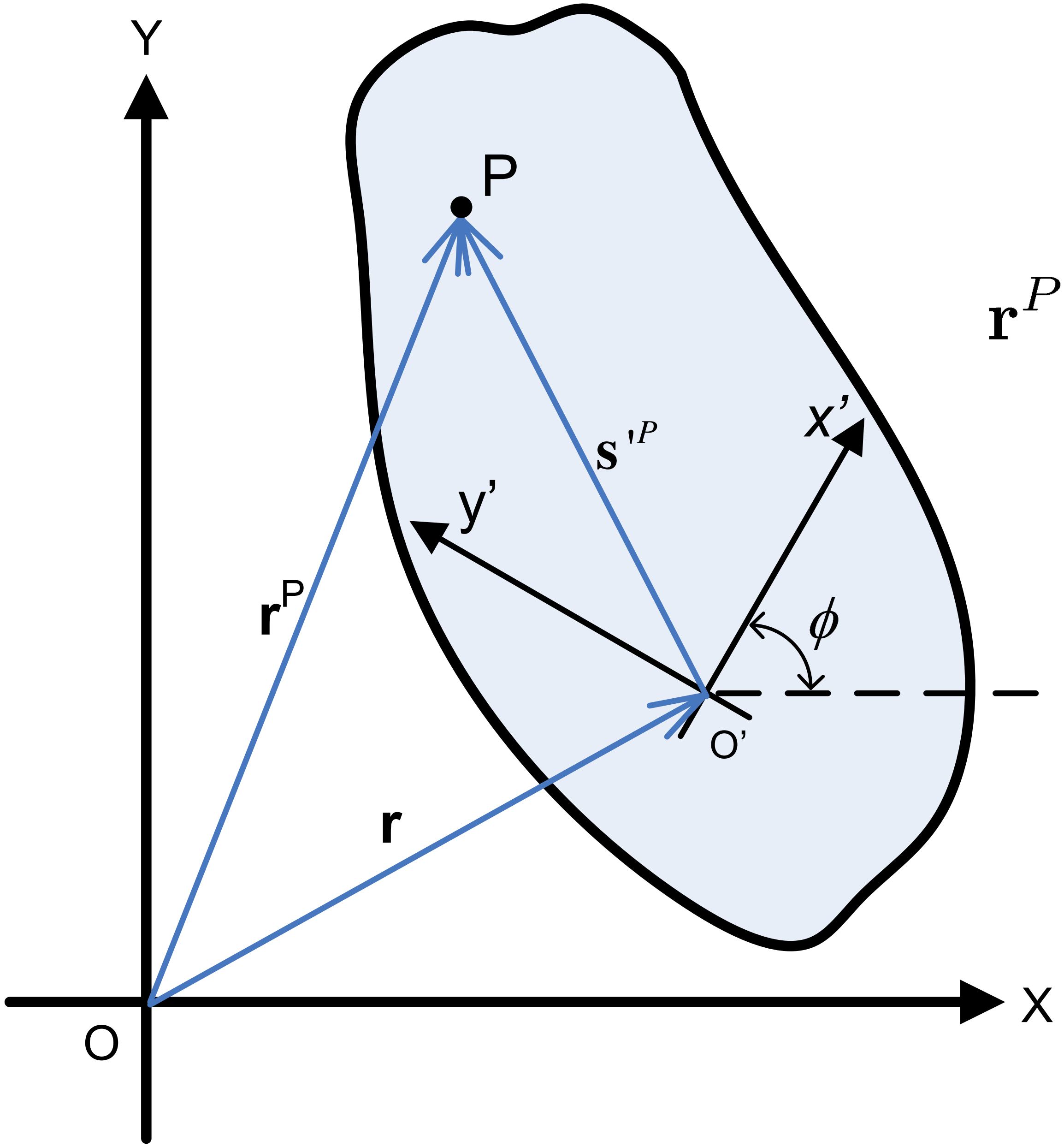


More on Body Kinematics

- A lot of ME451 is based on the ability to look at the location of one point P in two different reference frames: a local reference frame (LRF) and a global reference frame (GRF)
  - Local reference frame is typically fixed (rigidly attached) to a body that is moving in space
  - Global reference frame is the "world" reference frame: it's not moving, and serve as the universal reference frame

- In the LRF, the position of point P is described by $\mathbf{s}'^P$ (sometimes the notation used is $\bar{\mathbf{s}}^P$)
- In the GRF, the position of point P is described by $\mathbf{r}^P$ (see next slide)

ME451 Important Slide



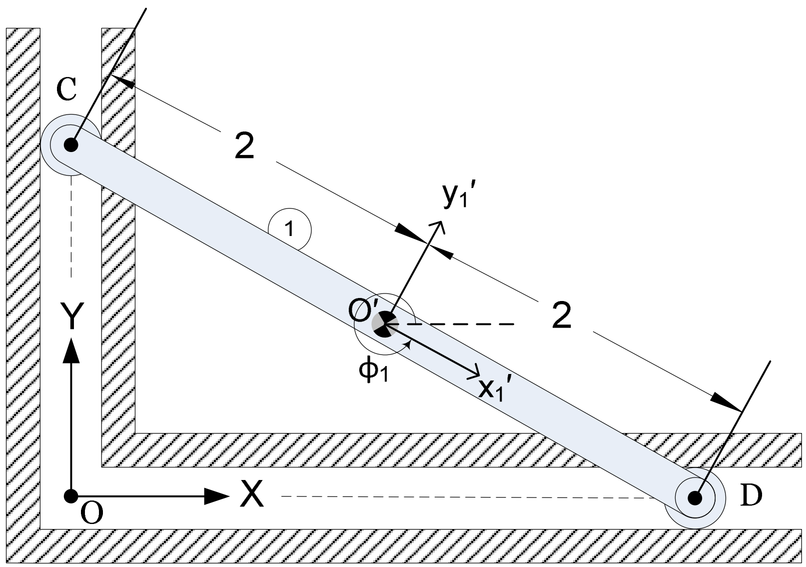
$$\mathbf{r}^P = \mathbf{r} + \mathbf{s}^P = \mathbf{r} + \mathbf{A}\mathbf{s}'^P$$

$$\mathbf{s}^P = \mathbf{A}\mathbf{s}'^P$$

$$\mathbf{A} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

Example

- The location of point O' in the OXY global RF is $[x,y]^T$. The orientation of the bar is described by the angle ϕ_1. Find the location of C and D expressed in the global reference frame as functions of x, y, and ϕ_1.



Matrix Review

Recall Notation Conventions

- A bold upper case letter denotes matrices
  - Example: **A**, **B**, etc.
- A bold lower case letter denotes a vector
  - Example: **v**, **s**, etc.
- A letter in italics format denotes a scalar quantity
  - Example: a, b_1

Matrix Review

- What is a matrix? A tableau of numbers ordered by row and column.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{\alpha}_1^T \\ \boldsymbol{\alpha}_2^T \\ \dots \\ \boldsymbol{\alpha}_m^T \end{bmatrix}$$

- Matrix addition:

$$\mathbf{A} = [a_{ij}], \qquad 1 \le i \le m, \qquad 1 \le j \le n$$

$$\mathbf{B} = [b_{ij}], \qquad 1 \le i \le m, \qquad 1 \le j \le n$$

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = [c_{ij}], \qquad c_{ij} = a_{ij} + b_{ij}$$

  - Addition is commutative

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

Matrix Multiplication

- Recall dimension constraints on matrices so that they can be multiplied:
  - # of columns of first matrix is equal to # of rows of second matrix

$$\mathbf{A} = [a_{ij}], \qquad \mathbf{A} \in \mathbb{R}^{m*n}$$

$$\mathbf{C} = [c_{ij}], \qquad \mathbf{C} \in \mathbb{R}^{n*p}$$

$$\mathbf{D} = \mathbf{A} \cdot \mathbf{C} = [d_{ij}], \qquad \mathbf{D} \in \mathbb{R}^{m*p}$$

$$d_{ij} = \sum_{k=1}^{n} a_{ik} c_{kj}$$

- Matrix multiplication is not commutative

- Distributivity of matrix multiplication with respect to matrix addition:

$$(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$$

Matrix-Vector Multiplication

- A column-wise perspective on matrix-vector multiplication (part of your HW)

$$\mathbf{A}\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \cdots \\ v_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \cdots \\ v_n \end{bmatrix} = \sum_{i=1}^{n} \mathbf{a}_i v_i$$

  - Example:

$$\mathbf{A}\mathbf{v} = \begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 3 & 1 & 1 \\ -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} \cdot (1) + \begin{bmatrix} 4 \\ 3 \\ 0 \\ 1 \end{bmatrix} \cdot (2) + \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix} \cdot (-1) + \begin{bmatrix} 0 \\ 1 \\ -1 \\ -2 \end{bmatrix} \cdot (1) = \begin{bmatrix} 7 \\ 8 \\ -3 \\ 1 \end{bmatrix}$$

- A row-wise perspective on matrix-vector multiplication:

$$\mathbf{A}\mathbf{v} = \begin{bmatrix} \boldsymbol{\alpha}_1^T \\ \boldsymbol{\alpha}_2^T \\ \cdots \\ \boldsymbol{\alpha}_m^T \end{bmatrix} \mathbf{v} = \begin{bmatrix} \boldsymbol{\alpha}_1^T \mathbf{v} \\ \boldsymbol{\alpha}_2^T \mathbf{v} \\ \cdots \\ \boldsymbol{\alpha}_m^T \mathbf{v} \end{bmatrix}$$

Orthogonal & Orthonormal Matrices

- Definition (**Q**, orthogonal matrix): a square matrix **Q** is orthogonal if the product $\mathbf{Q}^T\mathbf{Q}$ is a diagonal matrix

- Matrix **Q** is called orthonormal if it's orthogonal and also $\mathbf{Q}^T\mathbf{Q}=\mathbf{I}_n$
  - Note that in many cases people fail to make a distinction between an orthogonal and orthonormal matrix. We'll observe this distinction

- Note that if **Q** is an orthonormal matrix, then $\mathbf{Q}^{-1}=\mathbf{Q}^T$

- Example, orthonormal matrix:

$$\mathbf{Q} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Exercise

- Prove that the orientation **A** matrix is orthonormal

$$\mathbf{A} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

Remark:

On the Columns of an Orthonormal Matrix

- Assume **Q** is an orthonormal matrix

$$\mathbf{Q} \in \mathbb{R}^{n\times n} \qquad \mathbf{Q} = [\mathbf{q}_1, ..., \mathbf{q}_n] \quad \leftarrow \quad \text{orthonormal}$$

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I}_{n\times n} \qquad \Rightarrow \qquad \begin{bmatrix} \mathbf{q}_1^T \\ .. \\ \mathbf{q}_n^T \end{bmatrix} [\mathbf{q}_1, ..., \mathbf{q}_n] = \begin{bmatrix} \mathbf{q}_1^T\mathbf{q}_1 & .. & \mathbf{q}_1^T\mathbf{q}_n \\ .. & .. & .. \\ \mathbf{q}_n^T\mathbf{q}_1 & .. & \mathbf{q}_n^T\mathbf{q}_n \end{bmatrix} = \mathbf{I}_{n\times n}$$

$$\Downarrow$$

$$\mathbf{q}_i^T \cdot \mathbf{q}_j = \delta_{ij} = \begin{cases} 1 & if \quad i = j \\ 0 & if \quad i \neq j \end{cases}$$

- In other words, the columns of an orthonormal matrix have unit norm and are mutually perpendicular to each other

Matrix Review [Cntd.]

- Scaling of a matrix by a real number: scale each entry of the matrix

$$\alpha \cdot \mathbf{A} = \alpha \cdot [a_{ij}] = [\alpha \cdot a_{ij}]$$

  - Example:

$$(1.5) \cdot \begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 3 & 1 & 1 \\ -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 1.5 & 6 & 3 & 0 \\ 3 & 4.5 & 1.5 & 1.5 \\ -1.5 & 0 & 1.5 & -1.5 \\ 0 & 1.5 & -1.5 & -3 \end{bmatrix}$$

- Transpose of a matrix **A** dimension m×n: a matrix $\mathbf{B}=\mathbf{A}^T$ of dimension n×m whose (i,j) entry is the (j,i) entry of original matrix **A**

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 3 & 1 & 1 \\ -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & -2 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 4 & 3 & 0 & 1 \\ 2 & 1 & 1 & -1 \\ 0 & 1 & -1 & -2 \end{bmatrix}$$

Linear Independence of Vectors

- Definition: linear independence of a set of m vectors, $\mathbf{v}_1,\ldots,\mathbf{v}_m$:

$$\mathbf{v}_1, \ldots., \mathbf{v}_m \in \mathbb{R}^n$$

- The vectors are linearly independent if the following condition holds

$$\alpha_1 \mathbf{v}_1 + \ldots. + \alpha_m \mathbf{v}_m = \mathbf{0}_n \qquad \Rightarrow \qquad \alpha_1 = \ldots = \alpha_m = 0$$

- If a set of vectors are not linearly independent, they are called dependent
  - Example:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ -3 \\ -6 \end{bmatrix}$$

  - Note that $\mathbf{v}_1 - 2\mathbf{v}_2 - \mathbf{v}_3 = 0$

Matrix Rank

- Row rank of a matrix
  - Largest number of rows of the matrix that are linearly independent
  - A matrix is said to have full row rank if the rank of the matrix is equal to the number of rows of that matrix

- Column rank of a matrix
  - Largest number of columns of the matrix that are linearly independent

- NOTE: for each matrix, the row rank and column rank are the same
  - This number is simply called the rank of the matrix
  - It follows that

$$\mathrm{rank}(C) = \mathrm{rank}(C^T)$$

Matrix Rank, Example

- What is the row rank of the matrix **J**?

$$\mathbf{J} = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 4 & -2 & -2 & 1 \\ 0 & -4 & 0 & 1 \end{bmatrix}$$

- What is the rank of **J**?

Matrix Review [Cntd.]

- Symmetric matrix: a square matrix **A** for which **A**=**A**^{T}
- Skew-symmetric matrix: a square matrix **B** for which **B**=-**B**^{T}
- Examples:

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 3 \\ -1 & 3 & 4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 4 \\ -2 & -4 & 0 \end{bmatrix}$$

- Singular matrix: square matrix whose determinant is zero

$$\det(\mathbf{A}) = 0, \qquad \mathbf{A} \in \mathbb{R}^{n \times n}$$

- Inverse of a square matrix **A**: a matrix of the same dimension, called **A**$^{-1}$, that satisfies the following:

$$\mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{A}^{-1} = \mathbf{I}_n, \qquad \mathbf{A} \in \mathbb{R}^{n \times n}$$

Singular vs. Nonsingular Matrices

- Let **A** be a square matrix of dimension *n*. The following are equivalent:
  - $\mathbf{Ax} = \mathbf{b}$ has a unique solution for any $\mathbf{b} \in \mathbb{R}^n$.
  - $\mathbf{Ax} = \mathbf{b}$ has a solution for any $\mathbf{b} \in \mathbb{R}^n$.
  - $\mathbf{Ax} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$.
  - $\mathbf{A}^{-1}$ exists.
  - $\text{Determinant}(\mathbf{A}) \neq 0$.
  - $\text{rank}(\mathbf{A}) = n$.

Other Useful Formulas

[Pretty straightforward to prove true]

- If **A** and **B** are invertible, their product is invertible and

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

- Also,

$$\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$$

- For any two matrices **A** and **B** that can be multiplied

$$(\mathbf{AB})^{T} = \mathbf{B}^{T}\mathbf{A}^{T}$$

- For any three matrices **A**, **B**, and **C** that can be multiplied

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$$

Absolute (Cartesian) Generalized Coordinates

vs.

Relative Generalized Coordinates