ME451
Kinematics and Dynamics of Machine Systems

Review of Linear Algebra

2.1 through 2.4

Th, Sept. 08

Before we get started…

- Last time:
  - Syllabus
  - Quick overview of course
  - Starting discussion about vectors and their geometric representation

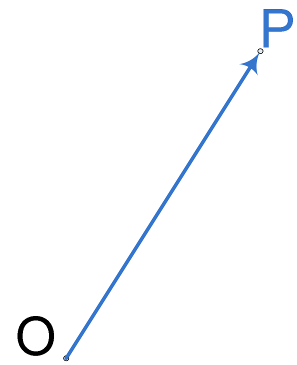
- HW Assigned:
  - Problems: 2.2.5, 2.2.8. 2.2.10
  - Problems out of Haug's book, available at class website
  - Due in one week

- Miscellaneous
  - Class website is up (http://sbel.wisc.edu/Courses/ME451/2011/index.htm)
  - Forum login credentials emailed to you by Mo EOB

Geometric Entities: Their Relevance

- Kinematics & Dynamics of systems of rigid bodies:
  - Requires the ability to describe the position, velocity, and acceleration of each rigid body in the system as functions of time
  - In the Euclidian 2D space, geometric vectors and 2X2 orthonormal matrices are extensively used to this end
    - Geometric vectors - used to locate points on a body or the center of mass of a rigid body
    - 2X2 orthonormal matrices - used to describe the orientation of a body

Geometric Vectors



- What is a "Geometric Vector"?
  - A quantity that has three attributes:
    - A support line (given by the blue line)
    - A direction along this line (from O to P)
    - A magnitude, ||OP||
  - Note that all geometric vectors are defined in relation to an origin **O**

- <u>IMPORTANT</u>:
  - Geometric vectors are entities that are independent of any reference frame

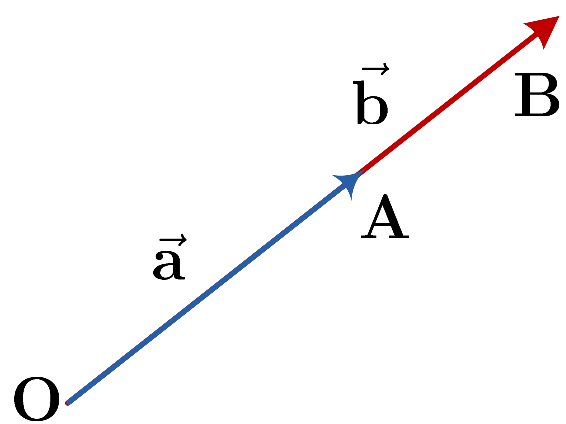
- ME451 deals <u>planar</u> kinematics and dynamics
  - We assume that all the vectors are defined in the 2D Euclidian space
  - A basis for the Euclidian space is any collection of two independent vectors

Geometric Vectors: Operations

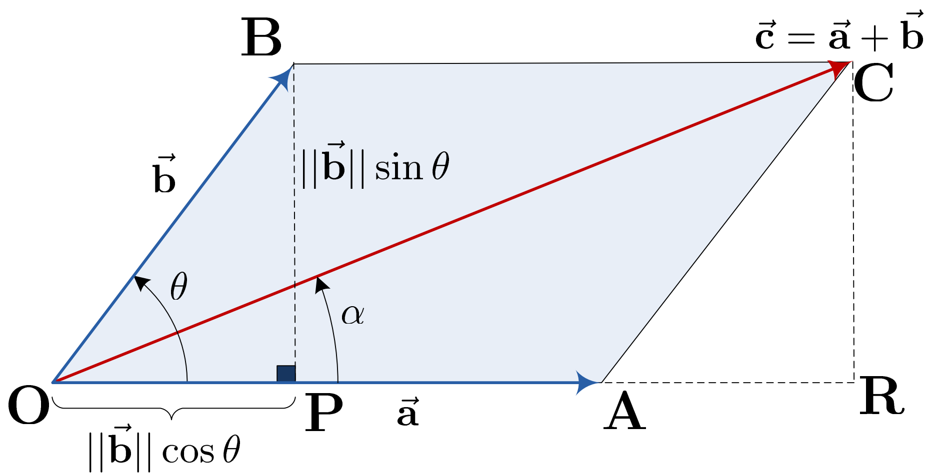
- What geometric vectors operations are <u>defined</u> out there?
  - Scaling by a scalar α
  - Addition of geometric vectors (the parallelogram rule)
  - One can measure the angle θ between two geometric vectors
  - Multiplication of two geometric vectors
    - The inner product rule (leads to a number)
    - The outer product rule (leads to a vector)
- A review these definitions follows over the next couple of slides

G. Vector Operation : Scaling by α

- By definition, scaling one geometric vector $\vec{\mathbf{a}}$ by a scalar $\alpha \neq 0$ leads to a new vector $\vec{\mathbf{b}} \equiv \alpha\vec{\mathbf{a}}$ that has the following three attributes:
  - $\vec{\mathbf{b}}$ has the same support line as the vector $\vec{\mathbf{a}}$
  - $\vec{\mathbf{b}}$ has the direction of $\vec{\mathbf{a}}$ if $\alpha > 0$, and opposite sense if $\alpha < 0$
  - The magnitude of $\vec{\mathbf{b}}$ is $b = |\alpha|a$
- Note that if $\alpha = 0$, then $\vec{\mathbf{b}}$ is the null vector.



G. Vector Operation: Addition of Two G. Vectors



- Sum of two vectors (definition)
  - Obtained by the parallelogram rule
- Operation is commutative
- Easy to visualize, pretty messy to summarize in an analytical fashion:

$$c = \sqrt{||\mathbf{OR}||^2 + ||\mathbf{RC}||^2} = \sqrt{(a + b\cos\theta)^2 + b^2 \sin^2\theta}$$

$$\alpha = \tan^{-1} \frac{b \sin\theta}{a + b\cos\theta}$$

G. Vector Operation: Angle Between Two G. Vectors

- Regarding the angle between two vectors, note that

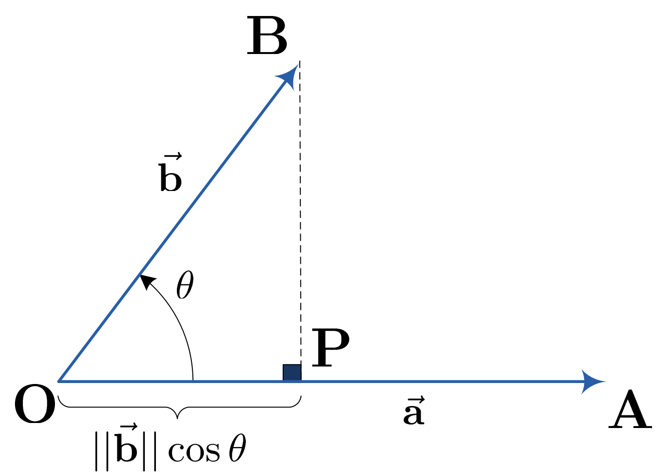
$$\theta(\vec{\mathbf{a}}, \vec{\mathbf{b}}) \neq \theta(\vec{\mathbf{b}}, \vec{\mathbf{a}}) \qquad \theta(\vec{\mathbf{a}}, \vec{\mathbf{b}}) = 2\pi - \theta(\vec{\mathbf{b}}, \vec{\mathbf{a}})$$

- Important: Angles are positive <u>counterclockwise</u>
  - This is why when measuring the angle between two vectors it's important what the first (start) vector is

G. Vector Operation: Inner Product of Two G. Vectors

- The product between the magnitude of the first geometric vector and the projection of the second vector onto the first vector

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = ||\vec{\mathbf{a}}|| \cdot ||\vec{\mathbf{b}}|| \cdot \cos(\vec{\mathbf{a}}, \vec{\mathbf{b}})$$



- Note that operation is commutative:

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \vec{\mathbf{b}} \cdot \vec{\mathbf{a}}$$

- Like the “dot product” of two algebraic vectors
  - NOTE: “inner product” and “outer product” is the nomenclature for geometric vector operations

Combining Basic G. Vector Operations

Using the definition of the geometric vector, the following four useful properties can be proved to be true

- P1 – The sum of geometric vectors is associative

$$\vec{\mathbf{a}} + (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = (\vec{\mathbf{a}} + \vec{\mathbf{b}}) + \vec{\mathbf{c}}$$

- P2 – Multiplication with a scalar is distributive with respect to the sum:

$$k \cdot (\vec{\mathbf{a}} + \vec{\mathbf{b}}) = (\vec{\mathbf{a}} + \vec{\mathbf{b}}) \cdot k = k \cdot \vec{\mathbf{a}} + k \cdot \vec{\mathbf{b}}$$

- P3 – The inner product is distributive with respect to sum:

$$\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} + \vec{\mathbf{a}} \cdot \vec{\mathbf{c}}$$

- P4:

$$\vec{\mathbf{b}}(\alpha + \beta) = (\alpha + \beta)\vec{\mathbf{b}} = \alpha \cdot \vec{\mathbf{b}} + \beta \cdot \vec{\mathbf{b}}$$

[PostedOnline]

Exercise, P3:

- Prove that inner product is distributive with respect to sum:

$$\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} + \vec{\mathbf{a}} \cdot \vec{\mathbf{c}}$$

Relying on Orthonormal Basis Vectors
(Introducing Reference Frames to Make Things Simpler)

- Geometric vectors:
  - Easy to visualize but cumbersome to work with
  - The major drawback: hard to manipulate
    - Was very hard to carry out simple operations (recall proving the distributive property on previous slide)
    - When it comes to computers, which are good at storing matrices and vectors, having to deal with a geometric entity is cumbersome
- We are about to address these drawbacks by introducing a pair of independent vectors (a basis for the 2D Euclidian space) in which we'll express any geometric vector

The Orthonormal Basis and the Corresponding Reference Frame

- Note that two independent vectors (a basis for 2D) define a Reference Frame (RF)

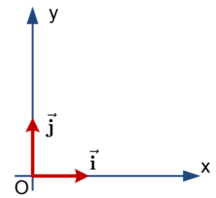
- In this class, to simplify our life, we use a set of two <u>orthonormal</u> unit vectors
  - These two vectors, $\vec{\mathbf{i}}$ and $\vec{\mathbf{j}}$, define the x and y directions of the RF

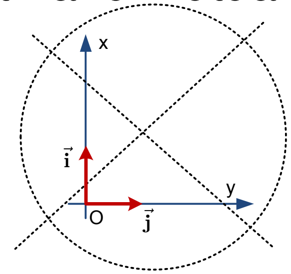
- A vector **a** can then be resolved into components a_x and a_y , along the axes *x* and *y* :

$$\vec{\mathbf{a}} = a_x \vec{\mathbf{i}} + a_y \vec{\mathbf{j}}$$

- Nomenclature: a_x and a_y are called the Cartesian components of the vector

- We're going to exclusively work with right hand mutually orthogonal RFs





Using a Basis Makes Life Simpler

- It is very easy now to quantify the result of various operations that involve geometric vectors

- Example: sum of two geometric vectors represented in a certain reference frame (basis)

$$\vec{\mathbf{a}} \xrightarrow{RF} a_x \vec{\mathbf{i}} + a_y \vec{\mathbf{j}} \qquad \vec{\mathbf{b}} \xrightarrow{RF} b_x \vec{\mathbf{i}} + b_y \vec{\mathbf{j}}$$

$$\vec{\mathbf{a}} + \vec{\mathbf{b}} \xrightarrow{RF} (a_x + b_x)\vec{\mathbf{i}} + (a_y + b_y)\vec{\mathbf{j}}$$

- Example: scaling a geometric vector by a scalar:

$$\vec{\mathbf{a}} \xrightarrow{RF} a_x \vec{\mathbf{i}} + a_y \vec{\mathbf{j}} \qquad \alpha\vec{\mathbf{a}} \xrightarrow{RF} \alpha a_x \vec{\mathbf{i}} + \alpha a_y \vec{\mathbf{j}}$$

Revisiting Geometric Vectors Operations

[given that we now have a Reference Frame]

- Since the angle between basis vectors is π/2, it's easy to see that:

$$\vec{\mathbf{i}} \cdot \vec{\mathbf{i}} = \vec{\mathbf{j}} \cdot \vec{\mathbf{j}} = 1 \qquad \vec{\mathbf{i}} \cdot \vec{\mathbf{j}} = \vec{\mathbf{j}} \cdot \vec{\mathbf{i}} = 0$$

- Also, it's easy to see that the projections ax and ay on the two axes are

$$a_x = \vec{\mathbf{a}} \cdot \vec{\mathbf{i}} \qquad a_y = \vec{\mathbf{a}} \cdot \vec{\mathbf{j}}$$

- Recall the distributive property of the dot product

$$\left(\vec{\mathbf{a}} + \vec{\mathbf{b}}\right) \cdot \vec{\mathbf{c}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{c}} + \vec{\mathbf{b}} \cdot \vec{\mathbf{c}}$$

- Based on the relation above, the following holds (expression for inner product when working in a reference frame):

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \left(a_x \vec{\imath} + a_y \vec{\jmath}\right) \cdot \left(b_x \vec{\imath} + b_y \vec{\jmath}\right) = a_x\ b_x + a_y\ b_y$$

Geometric Vectors: Loose Ends

- Given a vector $\vec{\mathbf{a}}$, the orthogonal vector $\vec{\mathbf{a}}^{\perp}$ is obtained as

$$\vec{\mathbf{a}} = a_x \vec{i} + a_y \vec{j} \qquad \Rightarrow \qquad \vec{\mathbf{a}}^{\perp} = -a_y \vec{i} + a_x \vec{j} \qquad \& \qquad \vec{\mathbf{a}} \cdot \vec{\mathbf{a}}^{\perp} = 0$$

- Length of a vector expressed using Cartesian coordinates:

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{a}} = ||\vec{\mathbf{a}}|| \cdot ||\vec{\mathbf{a}}|| = a_x a_x + a_y a_y \qquad \Rightarrow \qquad ||\vec{\mathbf{a}}|| = \sqrt{a_x^2 + a_y^2}$$

- Important Notation Convention:
  - Throughout this class, vectors/matrices are in bold font, scalars are not (most often they are in italics)

New Concept: Algebraic Vectors

- As we've seen, given a RF, an arbitrary geometric vector can be represented by a pair (a_x, a_y):

$$\vec{\mathbf{a}} = a_x \vec{\mathbf{i}} + a_y \vec{\mathbf{j}} \qquad \Leftrightarrow \qquad \vec{\mathbf{a}} \mapsto (a_x, a_y)$$

- It doesn't take too much imagination to associate to each geometric vector a two-dimensional algebraic vector:

$$\vec{\mathbf{a}} = a_x \vec{\mathbf{i}} + a_y \vec{\mathbf{j}} \qquad \Leftrightarrow \qquad \mathbf{a} = \begin{bmatrix} a_x \\ a_y \end{bmatrix}$$

- Note that I dropped the arrow on **a** to indicate that we are talking about an algebraic vector

Putting Things in Perspective…

- Step 1: We started with geometric vectors
- Step 2: We introduced a reference frame; i.e., an orthonormal basis
- Step 3: Relative to that reference frame each geometric vector is uniquely represented as a pair of scalars (the Cartesian coordinates)
- Step 4: We generated an algebraic vector whose two entries are provided by the pair above
  - This vector is the algebraic representation of the geometric vector
- It's easy to see that the algebraic representations of the basis vectors are

$$\vec{\mathbf{i}} \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \vec{\mathbf{j}} \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Fundamental Question:
How do G. Vector Ops. Translate into A. Vector Ops.?

- There is a straight correspondence between the operations
- Just a different representation of an old concept
  - Scaling a G. Vector $\Leftrightarrow$ Scaling of corresponding A. Vector
  - Adding two G. Vectors $\Leftrightarrow$ Adding the corresponding two A. Vectors
  - Inner product of two G. Vectors $\Leftrightarrow$ Dot Product of the two A. Vectors
    - We'll talk about outer product later
  - Measure the angle θ between two G. Vectors $\rightarrow$ uses inner product, so it is based on the dot product of the corresponding A. Vectors

Geometric Vector
and its representation in different
Reference Frames

- Recall that an algebraic vector is just a representation of a geometric vector in a particular reference frame (RF)

- Question: What if I now want to represent the same geometric vector in a different RF?
  - This is equivalent to a change of the basis used to represent the geometric vector

Algebraic Vector and Reference Frames

- Representing the same geometric vector in a different RF leads to the concept of Rotation Matrix **A**:
  - Getting the new coordinates, that is, representation of the **<u>same</u>** geometric vector in the new RF is as simple as multiplying the coordinates by the rotation matrix **A**:

$$\mathbf{s} = \mathbf{A}\mathbf{s}'$$

- NOTE 1: what is changed is the RF used for representing the vector, and not the underlying geometric vector
- NOTE 2: rotation matrix **A** is sometimes called "orientation matrix"

The Rotation Matrix A

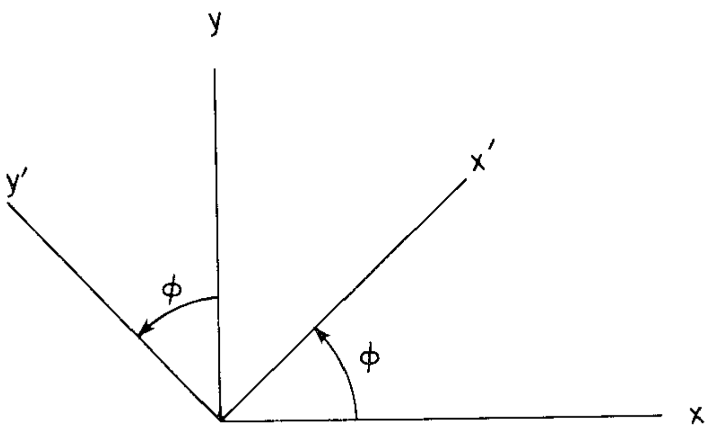


Figure 2.4.1 Two Cartesian reference frames.

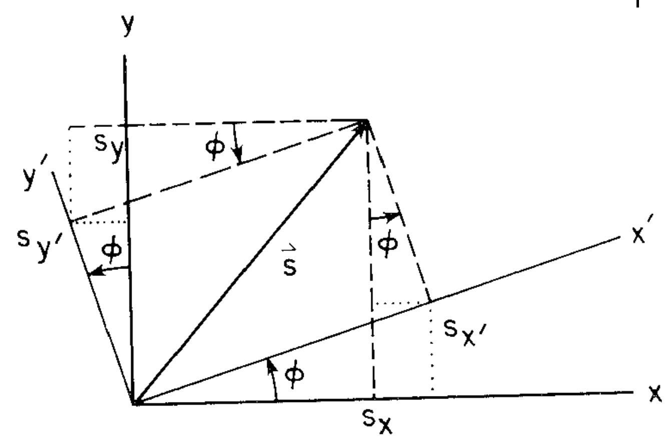


Figure 2.4.2 Vector $\vec{s}$ in two frames.

$$\mathbf{A} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

- Very important observation $\rightarrow$ the matrix **A** is orthonormal:

$$\mathbf{A}^T \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{A}^T = \mathbf{I}_{2\times 2}$$

Important Relation

- Expressing a given vector in one reference frame (local) in a different reference frame (global)

$$\mathbf{s} = \mathbf{A}\mathbf{s}'$$

Also called a change of base.