Example: Equation of Motion

$n$-body Slider-Crank Mechanism

![Diagram of a two-body slider-crank mechanism.](image)

Figure 6.3.2 Two-body slider–crank.

The eom for any mechanism assume the form:

$$M \ddot{q} + \Phi^T \lambda = \mathbf{Q}^e.$$  

For our mechanism, we need to identify the expression

$$\dot{q} = [n_1 \quad \phi_1 \quad \phi_2], \quad \mathbf{q}^e.$$  

$$\mathbf{M} = \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix}$$

As far as the constraints are concerned, we have:

- Revolute joint between Body 1 and ground.
- Revolute joint between Body 1 and Body 2.
- y absolute constraint for point D on Body 2.
\[ \Pi_1^D = \Pi_1 + A_1 S_1^D = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} c \phi_1 & -s \phi_1 \\ s \phi_1 & c \phi_1 \end{bmatrix} \begin{bmatrix} -p_1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 - p_1 c \phi_1 \\ y_1 + p_1 s \phi_1 \end{bmatrix} \]

\[ \Pi_2^C = \Pi_2 + A_2 S_2^C = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} c \phi_2 & -s \phi_2 \\ s \phi_2 & c \phi_2 \end{bmatrix} \begin{bmatrix} -p_2 \\ 0 \end{bmatrix} = \begin{bmatrix} x_2 - p_2 c \phi_2 \\ y_2 + p_2 s \phi_2 \end{bmatrix} \]

Then,

\[ \Phi(\eta, t) = \begin{bmatrix} x_1 - p_1 c \phi_1 \\ y_1 - p_1 s \phi_1 \\ x_1 + p_1 c \phi_1 - x_2 - p_2 c \phi_2 \\ y_1 + p_1 s \phi_1 - y_2 + p_2 s \phi_2 \\ y_2 + p_2 s \phi_2 \end{bmatrix} = 0 \]

\[ \Phi_q = \begin{bmatrix} 1 & 0 & p_1 c \phi_1 & 0 & 0 & 0 \\ 0 & 1 & -p_1 c \phi_1 & 0 & 0 & 0 \\ 1 & 0 & -p_2 s \phi_2 & 0 & 0 & -p_2 c \phi_2 \\ 0 & 1 & p_1 c \phi_1 & 0 & -1 & p_2 c \phi_2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \]
Applied forces:

\[ \Phi_1^A = \begin{bmatrix} -m_1 g \\ n \end{bmatrix} \quad \Phi_2^A = \begin{bmatrix} 0 \\ -m_2 g \end{bmatrix} \]

Note: \( n \) is a given applied torque.

Since we have 5 constraints, we'll have 5 Lagrange multipliers:

\[ \lambda \equiv \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{bmatrix} \]

The equations of motion then assume the expression:

\[
\begin{bmatrix}
\Phi_1 & 0 & 0 & 0 & 0 \\
0 & \Phi_1 & 0 & 0 & 0 \\
0 & 0 & \Phi_1 & 0 & 0 \\
0 & 0 & 0 & \Phi_1 & 0 \\
0 & 0 & 0 & 0 & \Phi_1 \\
\end{bmatrix}
\begin{bmatrix}
\ddot{x}_1 \\
\ddot{x}_2 \\
\ddot{x}_3 \\
\ddot{x}_4 \\
\ddot{x}_5 \\
\end{bmatrix}
+ \begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\lambda_4 \\
\lambda_5 \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

This needs to be coupled with the acceleration equation to lead to a linear system of equations in the unknowns \( \ddot{q} \) and \( \lambda \). The acceleration equation is

\[ \ddot{q} \cdot \ddot{q} = 0 \]

For our problem \( \ddot{q} \) is computed below.
The constrained form will then assume the form:
Note that this is slightly different than what's in the book due to the set of constraints that I worked with. They are equivalent and equally good but they are different in form.