Example: Equation of Motion ~ Slider Crank Mechanism ~

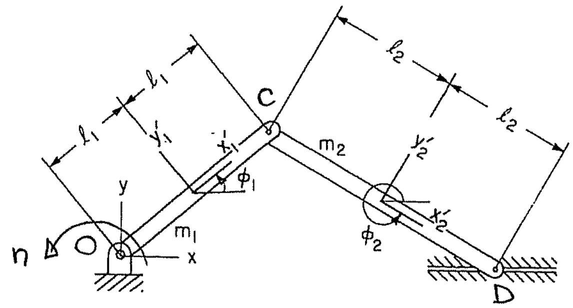

Figure 6.3.2 Two-body slider–crank.

The EOM for any mechanism assume the form:

$$M\ddot{q} + \Phi_q^T \lambda = Q^A.$$

For our mechanism, we need to identify the expression of M, Φ, Φ_q, Q^A.

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ \phi_1 \\ r_2 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ \phi_1 \\ x_2 \\ y_2 \\ \phi_2 \end{bmatrix}$$

$$M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} = \begin{bmatrix} m_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & J_1' & 0 & 0 & 0 \\ 0 & 0 & 0 & m_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & J_2' \end{bmatrix}$$

As far as the constraints are concerned, we have

- Revolute joint between Body 1 and ground
- Revolute joint between Body 1 and Body 2.
- y absolute constraint for point D on body 2

$$r_1^a = r_1 + A_1 s_1'^a = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \end{bmatrix}\begin{bmatrix} -l_1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 - l_1 c\phi_1 \\ y_1 - l_1 s\phi_1 \end{bmatrix}$$

$$r_1^c = r_1 + A_1 s_1'^c = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \end{bmatrix}\begin{bmatrix} l_1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 + l_1 c\phi_1 \\ y_1 + l_1 s\phi_1 \end{bmatrix}$$

$$r_2^c = r_2 + A_2 s_2'^c = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} c\phi_2 & -s\phi_2 \\ s\phi_2 & c\phi_2 \end{bmatrix}\begin{bmatrix} -l_2 \\ 0 \end{bmatrix} = \begin{bmatrix} x_2 - l_2 c\phi_2 \\ y_2 - l_2 s\phi_2 \end{bmatrix}$$

$$r_2^D = r_2 + A_2 s_2'^D = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} c\phi_2 & -s\phi_2 \\ s\phi_2 & c\phi_2 \end{bmatrix}\begin{bmatrix} l_2 \\ 0 \end{bmatrix} = \begin{bmatrix} x_2 + l_2 c\phi_2 \\ y_2 + l_2 s\phi_2 \end{bmatrix}$$

Then,

$$\phi(q,t) = \begin{bmatrix} x_1 - l_1 c\phi_1 \\ y_1 - l_1 s\phi_1 \\ x_1 + l_1 c\phi_1 - x_2 + l_2 c\phi_2 \\ y_1 + l_1 s\phi_1 - y_2 + l_2 s\phi_2 \\ y_2 + l_2 s\phi_2 \end{bmatrix} = 0$$

$$\phi_q = \begin{bmatrix} 1 & 0 & l_1 s\phi_1 & 0 & 0 & 0 \\ 0 & 1 & -l_1 c\phi_1 & 0 & 0 & 0 \\ 1 & 0 & -l_1 s\phi_1 & -1 & 0 & -l_2 s\phi_2 \\ 0 & 1 & l_1 c\phi_1 & 0 & -1 & l_2 c\phi_2 \\ 0 & 0 & 0 & 0 & 1 & l_2 c\phi_2 \end{bmatrix}$$

Applied forces:

$$Q_1^A = \begin{bmatrix} 0 \\ -m_1 g \\ n \end{bmatrix} \qquad Q_2^A = \begin{bmatrix} 0 \\ -m_2 g \\ 0 \end{bmatrix}$$

Note: n is a given applied torque.

Since we have 5 constraints, we'll have 5 Lagrange multipliers:

$$\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{bmatrix}$$

The equations of motion then assume the expression:

$$\begin{bmatrix} m_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & J_1' & 0 & 0 & 0 \\ 0 & 0 & 0 & m_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & J_2' \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\phi}_1 \\ \ddot{x}_2 \\ \ddot{y}_2 \\ \ddot{\phi}_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ l_1 s\phi_1 & -l_1 c\phi_1 & -l_1 s\phi_1 & l_1 c\phi_1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -l_2 s\phi_2 & l_2 c\phi_2 & l_2 c\phi_2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{bmatrix} = \begin{bmatrix} 0 \\ -m_1 g \\ n \\ 0 \\ -m_2 g \\ 0 \end{bmatrix}$$

This needs to be coupled with the acceleration equation to lead to a linear system of equations in the unknowns $\ddot{q}$ and λ. The acceleration equation is

$$\Phi_q \cdot \ddot{q} = \gamma .$$

For our problem γ is computed below.

$$\dot{\Phi} = \begin{bmatrix} \dot{x}_1 + l_1 \dot{\phi}_1 s\phi_1 \\ \dot{y}_1 - l_1 \dot{\phi}_1 \, c\phi_1 \\ \dot{x}_1 - l_1 \dot{\phi}_1 s\phi_1 - \dot{x}_2 - l_2 \dot{\phi}_2 s\phi_2 \\ \dot{y}_1 + l_1 \dot{\phi}_1 c\phi_1 - \dot{y}_2 + l_2 \dot{\phi}_2 c\phi_2 \\ \dot{y}_2 + l_2 \dot{\phi}_2 c\phi_2 . \end{bmatrix} = 0$$

$$\ddot{\Phi} = \begin{bmatrix} \ddot{x}_1 + \ddot{\phi}_1 l_1 s\phi_1 + l_1 \dot{\phi}_1^2 c\phi_1 \\ \ddot{y}_1 - \ddot{\phi}_1 l_1 c\phi_1 + l_1 \dot{\phi}_1^2 s\phi_1 \\ \ddot{x}_1 - \ddot{\phi}_1 l_1 s\phi_1 - l_1 \dot{\phi}_1^2 c\phi_1 - \ddot{x}_2 - l_2 \ddot{\phi}_2 s\phi_2 - l_2 \dot{\phi}_2^2 c\phi_2 \\ \ddot{y}_1 + \ddot{\phi}_1 l_1 c\phi_1 - l_1 \dot{\phi}_1^2 s\phi_1 - \ddot{y}_2 + \ddot{\phi}_2 l_2 c\phi_2 - l_2 \dot{\phi}_2^2 s\phi_2 \\ \ddot{y}_2 + \ddot{\phi}_2 l_2 c\phi_2 - l_2 \dot{\phi}_2^2 s\phi_2 \end{bmatrix} = 0$$

$$\Rightarrow \Phi_q \cdot \ddot{q} = \underbrace{\begin{bmatrix} - l_1 \dot{\phi}_1^2 c\phi_1 \\ - l_1 \dot{\phi}_1^2 s\phi_1 \\ l_1 \dot{\phi}_1^2 c\phi_1 + l_2 \dot{\phi}_2^2 c\phi_2 \\ l_1 \dot{\phi}_1^2 s\phi_1 + l_2 \dot{\phi}_2^2 s\phi_2 \\ l_2 \dot{\phi}_2^2 s\phi_2 \end{bmatrix}}_{\gamma}$$

The constrained EOM will then assume the form:

$$
\left[\begin{array}{cccccc|ccccc}
m_1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & m_1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & J_1' & 0 & 0 & 0 & l_1 s\phi_1 & -l_1 c\phi_1 & -l_1 s\phi_1 & l_1 c\phi_1 & 0 \\
0 & 0 & 0 & m_2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & m_2 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & J_2' & 0 & 0 & -l_2 s\phi_2 & l_2 c\phi_2 & l_2 c\phi_2 \\
\hline
1 & 0 & l_1 s\phi_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -l_1 c\phi_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -l_1 s\phi_1 & -1 & 0 & -l_2 s\phi_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & l_1 c\phi_1 & 0 & -1 & l_2 c\phi_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & l c\phi_2 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\left[\begin{array}{c}
\ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\phi}_1 \\ \ddot{x}_2 \\ \ddot{y}_2 \\ \ddot{\phi}_2 \\ \hline \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5
\end{array}\right]
=
\left[\begin{array}{c}
0 \\ -m_1 g \\ n \\ 0 \\ -m_2 g \\ 0 \\ -l_1 \dot{\phi}_1^2 c\phi_1 \\ -l_1 \dot{\phi}_1^2 s\phi_1 \\ l_1 \dot{\phi}_1^2 c\phi_1 + l_2 \dot{\phi}_2^2 c\phi_2 \\ l_1 \dot{\phi}_1^2 s\phi_1 + l_2 \dot{\phi}_2^2 s\phi_2 \\ l_2 \dot{\phi}_2^2 s\phi_2
\end{array}\right]
$$

Note that this is slightly different than what's in the book due to the set of constraints ϕ that I worked with. They are equivalent and equally good but they are different in form.