ME451
Kinematics and Dynamics of Machine Systems

Dynamics of Planar Systems

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Elements of the Numerical Solution of the Dynamics Problem
Chapter 7

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Before we get started…

- Last Time
  - Numerical solution of IVPs
  - Discretization schemes discussed:
    - Explicit: Forward Euler – expeditious but small stability region
    - Implicit: Backward Euler, BDF – requires solution of nonlinear system, typically much more stable than explicit methods

- Today
  - Look at a couple of examples for numerical solution of IVPs
  - Introduce the most basic approach for finding an approximation of the solution of the constrained equations of motion
    - We'll rely on the so called Newmark method

- Miscellaneous
  - Take Home Exam posted online. Due the night before the Final Exam
    - Draws on today's and Th's material

Example

[Solving IVP with Backward Euler]

- Use Backward Euler with a step-size Δt=0.1 to solve the following IVP:

$$\begin{cases} \dot{y} & = & -0.1y + \sin t \\ y(0) & = & 0 \end{cases}$$

- Reason for looking at this example: understand that when using an implicit integration formula (such as Backward Euler) upon discretization you have to solve an algebraic problem

- Recall that "discretization" is the process of transforming the continuum problem into a discrete problem
  - Helps us get the values of the unknown function at the discrete grid points
  - It is the reason why you need an integration formula (also called discretization formula)

Example

[Solving IVP with Backward Euler]

- This example shows how things can get tricky with implicit integrators

- Solve the following IVP using Backward Euler with a step-size Δt=0.1:

$$\left\{ \begin{array}{rcl} \dot{y} & = & -y^2 \\ y(0) & = & 2.4 \end{array} \right.$$

Example

[Solving IVP with Backward Euler]

- This example shows why using an implicit integrator brings into the picture the Newton-Raphson method

- Solve the following IVP using Backward Euler with a step-size Δt=0.1:

$$\begin{cases} \dot{y} & = & \sin(y) \\ y(0) & = & 0 \end{cases}$$

Conclusions, Implicit Integration

- For stiff IVPs, Implicit Integration is the way to go
  - It allows for integration step-sizes that might be orders of magnitude higher

- However, for most real-life IVP, an implicit integrator upon discretization leads to another nasty problem
  - You have to find the solution of a nonlinear algebraic problem
  - This brings into the picture the Newton-Raphson method (and its variants)
    - You have to deal with providing a starting point, computing the sensitivity matrix, etc.

End Numerical Methods for IVPs

Begin Numerical Methods for DAEs

Summary of the Lagrange form of the Constrained Equations of Motion

- Equations of Motion: $\mathbf{M}\ddot{\mathbf{q}} + \mathbf{\Phi}_{\mathbf{q}}^{T}\lambda = \mathbf{Q}^{A}$

- Position Constraint Equations: $\mathbf{\Phi}(\mathbf{q}, t) = \mathbf{0}$

- Velocity Constraint Equations: $\mathbf{\Phi}_{\mathbf{q}}\dot{\mathbf{q}} = \nu$

- Acceleration Constraint Equations: $\mathbf{\Phi}_{\mathbf{q}}\ddot{\mathbf{q}} = \gamma$

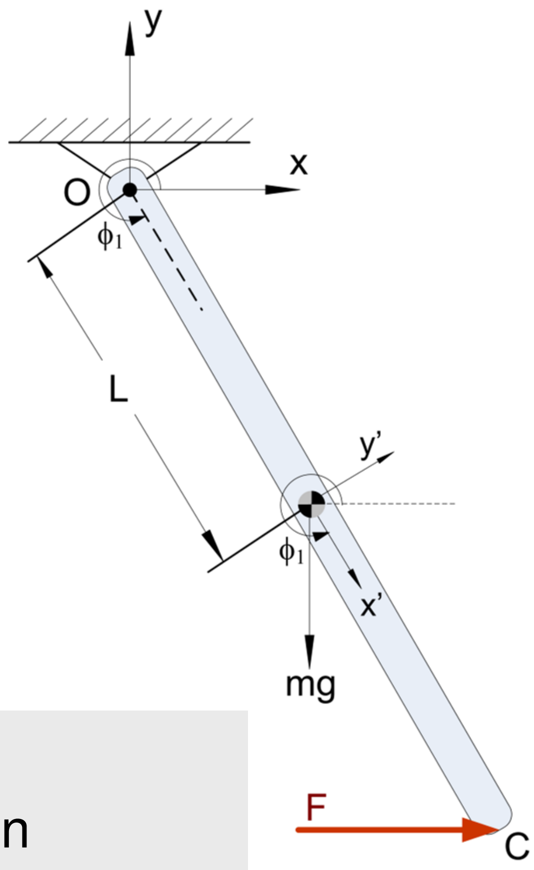
The Most Important Slide of ME451

What’s special about ME451 problems?

- Looking at the previous slide there are several things that make the ME451 dynamics problem challenging:
  - The problem is **not in standard form** $\dot{y} = f(t, y)$
  - Moreover, our problem is **not a first order** Ordinary Differential Equation (ODE) problem
    - Rather, it’s a second order ODE problem, due to form of the equations of motion (contain the second time derivative of the positions)
  - In the end, it’s **not even an ODE** problem
    - The unknown function **q**(t); that is, the position of the mechanism, is the solution of a second order ODE problem (first equation previous slide) but it must also satisfy a set of kinematic constraints at position, velocity, and acceleration levels, which are formulated as a bunch of algebraic equations
      - To conclude, you have to solve a set of differential-algebraic equations (DAEs)
      - DAEs are much tougher to solve than ODEs
      - This lecture is about using a numerical method to solve the DAEs of multibody dynamics

Example: Find the time evolution of the pendulum

- Simple Pendulum:
  - Mass 20 kg
  - Length L=2 m
  - Force acting at tip of pendulum
    - $F = 30 \sin(2\pi\ t)$ [N]
  - Although not shown in the picture, assume RSDA element in revolute joint
    - $k = 45$ [Nm/rad] & $\theta_0=3\pi/2$
    - $c = 10$ [N/s]
  - ICs: hanging down, starting from rest



- Stages of the procedure (three):
  - Stage 1: Derive constrained equations of motion
  - Stage 2: Indicate initial conditions (ICs)
  - Stage 3: Apply numerical integration algorithm to discretize DAE problem and turn into algebraic problem

Example: Simple Pendulum

Stage 1: Deriving EOM (see also handout)

Stage 2: Indicate Initial Conditions (ICs)

Algorithm for Resolving Dynamics of Mechanisms

- If you have the EOM and ICs, how do you go about solving the problem?

- This is a research topic in itself

- We'll cover one of the simplest algorithm possible
  - Based on Newmark's integration formulas
    - That is, we are going to use Newmark's formulas to discretize our index 3 DAE of constrained multibody dynamics

Solution Strategy

[Important Slide]

- This slide explains the strategy through which the numerical solution; i.e., an approximation of the actual solution of the dynamics problem, is produced

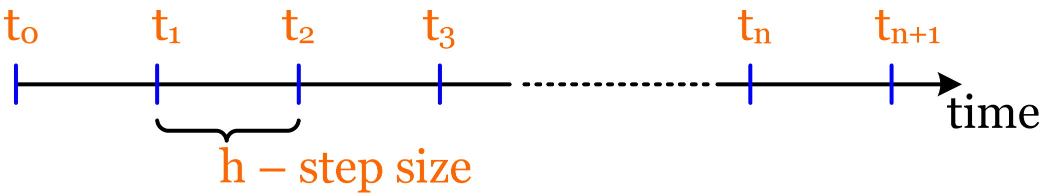
- Stage 1: two integration formulas (called Newmark in our case) are used to **express the positions and velocities as functions of accelerations**
  - These are also called “discretization formulas”

- Stage 2: everywhere in the constrained equations of motion, the positions and velocities are **replaced** using the discretization formulas and expressed in terms of the acceleration
  - This is the most important step, since through this “discretization” the **differential problem is transformed into an algebraic problem**
    - The algebraic problem, which effectively amounts to solving a nonlinear system, is approached using Newton’s method (so again, we need to produce a Jacobian)

- Stage 3: **solve a nonlinear system** to find the acceleration and Lagrange multipliers

Quick Overview, Newmark Integration Formulas

- Newmark method (N.M. Newmark – 1957)
  - Goal: find the positions, velocities, accelerations and Lagrange multipliers on a grid of time points; i.e., at t_0, t_1, etc.



  - Newmark's method: integrate directly *second* order equations of motion:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{F}(t) \quad \Longleftrightarrow \quad \mathbf{M}\ddot{\mathbf{q}}_{n+1} + \mathbf{C}\dot{\mathbf{q}}_{n+1} + \mathbf{K}\mathbf{q}_{n+1} = \mathbf{F}(t_{n+1})$$

  - Newmark's Method: relies on a set of "discretization" or "integration" formulas that relate *position to acceleration* and *velocity to acceleration*:

$$\mathbf{q}_{n+1} = \mathbf{q}_n + h\dot{\mathbf{q}}_n + \frac{h^2}{2}\left[(1-2\beta)\,\ddot{\mathbf{q}}_n + 2\beta\ddot{\mathbf{q}}_{n+1}\right]$$

$$\dot{\mathbf{q}}_{n+1} = \dot{\mathbf{q}}_n + h\left[(1-\gamma)\,\ddot{\mathbf{q}}_n + \gamma\ddot{\mathbf{q}}_{n+1}\right]$$

Newmark (Cntd.)

- Newmark Method
  - Accuracy: 1st Order
  - Initially introduced to deal with linear transient Finite Element Analysis
  - Stability: Very good stability properties
  - Choose β=0.3025, and γ=0.6 (these are two parameters that control the behavior of the method)

- If we write the equation of motion at each time t_{n+1} one gets

$$\mathbf{M}\ddot{\mathbf{q}}_{n+1} + \mathbf{C}\dot{\mathbf{q}}_{n+1} + \mathbf{K}\mathbf{q}_{n+1} = \mathbf{F}(t_{n+1})$$

- Now is the time to replace $\mathbf{q}_{n+1}$ and $\dot{\mathbf{q}}_{n+1}$ with the discretization formulas (see previous slide)
- You end up with an algebraic problem in which the unknown is exclusively the acceleration $\ddot{\mathbf{q}}_{n+1}$

The Problem at Hand

- Our rigid multibody dynamics problem is slightly more complicated than the Linear Finite Element problem used to introduce Newmark's discretization formulas
  - More complicated since we have some constraints that the solution must satisfy
  - We also have to deal with the Lagrange multipliers that come along with these constraints (from a physical perspective remember that they help enforce satisfaction of the constraints)

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{F}(t)$$

Linear Finite Element dynamics problem

$$\begin{cases} \mathbf{M}\ddot{\mathbf{q}} + \mathbf{\Phi}_{\mathbf{q}}^{T}(\mathbf{q})\lambda - \mathbf{Q}^{A}(\dot{\mathbf{q}}, \mathbf{q}, t) = \mathbf{0} \\ \mathbf{\Phi}\left(\mathbf{q}, t\right) = \mathbf{0} \end{cases}$$

Nonlinear multibody dynamics problem. Newmark algorithm still works as before, problem is slightly messier…

Quantities of Interest…

- Generalized accelerations: $\ddot{\mathbf{q}}$
- Generalized velocities: $\dot{\mathbf{q}}$
- Generalized positions: $\mathbf{q}$
- Reaction Forces: λ

All these quantities are functions of time (they change in time)

Stage 3: The Discretization of the Constrained Equations of Motion

- The Discretized Equations Solved at each Time-Step t_{n+1}:

$$\begin{cases} \mathbf{M}\ddot{\mathbf{q}}_{n+1} + \mathbf{\Phi}_{\mathbf{q}}^{T}(\mathbf{q}_{n+1})\lambda_{n+1} - \mathbf{Q}^{A}(t_{n+1}, \mathbf{q}_{n+1}, \dot{\mathbf{q}}_{n+1}) = \mathbf{0} \\ \frac{1}{\beta h^2}\mathbf{\Phi}\left(\mathbf{q}_{n+1}, t_{n+1}\right) = \mathbf{0} \end{cases}$$

- Above you see $\mathbf{q}_{n+1}$ and $\dot{\mathbf{q}}_{n+1}$, but remember that they are functions of the accelerations:

$$\mathbf{q}_{n+1} = \mathbf{q}_n + h\dot{\mathbf{q}}_n + \frac{h^2}{2}\left[(1-2\beta)\,\ddot{\mathbf{q}}_n + 2\beta\ddot{\mathbf{q}}_{n+1}\right]$$

$$\dot{\mathbf{q}}_{n+1} = \dot{\mathbf{q}}_n + h\left[(1-\gamma)\,\ddot{\mathbf{q}}_n + \gamma\ddot{\mathbf{q}}_{n+1}\right]$$

Again, these are Newmark's formulas that express the generalized position and velocity as functions of generalized accelerations

The Discretization of the Constrained Equations of Motion (Cntd.)

- Remarks on the discretization and its outcome:
  - Our unknowns are the accelerations and the Lagrange multipliers
  - The number of unknowns is equal to the number of equations
  - The equations that you have to solve now are algebraic and nonlinear
    - Differential problem has been transformed into an algebraic one
      - The new problem: find acceleration and Lagrange multipliers that satisfy

$$\Psi(\ddot{\mathbf{q}}_{n+1},\lambda_{n+1}) \equiv \begin{bmatrix} \mathbf{M}\ddot{\mathbf{q}}_{n+1} + \mathbf{\Phi}_{\mathbf{q}}^{T}(\mathbf{q}_{n+1})\lambda_{n+1} - \mathbf{Q}^{A}(t_{n+1},\mathbf{q}_{n+1},\dot{\mathbf{q}}_{n+1}) \\ \frac{1}{\beta h^2}\mathbf{\Phi}\left(\mathbf{q}_{n+1},t_{n+1}\right) \end{bmatrix} = \mathbf{0}$$

    - You have to use Newton’s method
      - This calls for the Jacobian of the nonlinear system of equations (chain rule will be used to simplify calculations)
      - This looks exactly like what we had to do when we dealt with the Kinematics analysis of a mechanism (there we solved $\mathbf{\Phi}(\mathbf{q},t)=0$ to get the positions $\mathbf{q}$)

[Three-Slide Detour]

Chain Rule for Computing Partial Derivatives

- Define the function Π as

$$\mathbf{\Pi}(\ddot{\mathbf{q}}_{n+1}, \dot{\mathbf{q}}_{n+1}, \mathbf{q}_{n+1}) = \mathbf{M}\ddot{\mathbf{q}}_{n+1} + \mathbf{\Phi}_{\mathbf{q}}^{T}(\mathbf{q}_{n+1})\lambda_{n+1} - \mathbf{Q}^{A}(t_{n+1}, \mathbf{q}_{n+1}, \dot{\mathbf{q}}_{n+1})$$

- Note that in fact Π is only a function of the acceleration
  - Based on Newmark's formulas, the velocity depends on acceleration and the position as well depends on acceleration.
- Therefore, I can define this new function Ω and clearly indicate that it only depends on acceleration:

$$\mathbf{\Omega}(\ddot{\mathbf{q}}_{n+1}) = \mathbf{\Pi}(\ddot{\mathbf{q}}_{n+1},\;\; \dot{\mathbf{q}}_{n+1}(\ddot{\mathbf{q}}_{n+1}),\;\; \mathbf{q}_{n+1}(\ddot{\mathbf{q}}_{n+1}))$$

[Three-Slide Detour]

Chain Rule for Computing Partial Derivatives

- I'm interested in the sensitivity of Ω w.r.t. the acceleration. To this end, apply the chain rule:

$$\frac{\partial \boldsymbol{\Omega}}{\partial \ddot{\mathbf{q}}_{n+1}} = \frac{\partial \boldsymbol{\Pi}}{\partial \ddot{\mathbf{q}}_{n+1}} + \frac{\partial \boldsymbol{\Pi}}{\partial \mathbf{q}_{n+1}} \cdot \frac{\partial \mathbf{q}_{n+1}}{\partial \ddot{\mathbf{q}}_{n+1}} + \frac{\partial \boldsymbol{\Pi}}{\partial \dot{\mathbf{q}}_{n+1}} \cdot \frac{\partial \dot{\mathbf{q}}_{n+1}}{\partial \ddot{\mathbf{q}}_{n+1}}$$

- Note that:

$$\frac{\partial \mathbf{q}_{n+1}}{\partial \ddot{\mathbf{q}}_{n+1}} = \beta h^2 \mathbf{I}_{n\times n} \qquad \frac{\partial \dot{\mathbf{q}}_{n+1}}{\partial \ddot{\mathbf{q}}_{n+1}} = \gamma h \mathbf{I}_{n\times n}$$

- Therefore, I conclude that the sensitivity of Ω w.r.t. the generalized acceleration will be computed as

$$\frac{\partial \boldsymbol{\Omega}}{\partial \ddot{\mathbf{q}}_{n+1}} = \frac{\partial \boldsymbol{\Pi}}{\partial \ddot{\mathbf{q}}_{n+1}} + \frac{\partial \boldsymbol{\Pi}}{\partial \mathbf{q}_{n+1}} \beta h^2 \mathbf{I}_{n\times n} + \frac{\partial \boldsymbol{\Pi}}{\partial \dot{\mathbf{q}}_{n+1}} \gamma h \mathbf{I}_{n\times n}$$

- Equivalently,

$$\boxed{\frac{\partial \boldsymbol{\Omega}}{\partial \ddot{\mathbf{q}}_{n+1}} = \frac{\partial \boldsymbol{\Pi}}{\partial \ddot{\mathbf{q}}_{n+1}} + \beta h^2 \frac{\partial \boldsymbol{\Pi}}{\partial \mathbf{q}_{n+1}} + \gamma h \frac{\partial \boldsymbol{\Pi}}{\partial \dot{\mathbf{q}}_{n+1}}}$$

[Three-Slide Detour]

Handling the Kinematic Constraints

- The focus here is on computing the sensitivity of the constraint equations with respect to the accelerations
- Note that I chose to scale the constraint equations by the factor 1/βh^2. This is ok since both β and h are constants

- The question is as follows:

$$\frac{\partial[\frac{1}{\beta h^2}\mathbf{\Phi}(\mathbf{q}_{n+1})]}{\partial \ddot{\mathbf{q}}_{n+1}} \quad = \quad ?$$

- Recall that

$$\frac{\partial \mathbf{q}_{n+1}}{\partial \ddot{\mathbf{q}}_{n+1}} = \beta h^2 \mathbf{I}_{n\times n} \qquad \frac{\partial \dot{\mathbf{q}}_{n+1}}{\partial \ddot{\mathbf{q}}_{n+1}} = \gamma h \mathbf{I}_{n\times n}$$

- Then, using the chain rule

$$\frac{\partial[\frac{1}{\beta h^2}\mathbf{\Phi}(\mathbf{q}_{n+1})]}{\partial \ddot{\mathbf{q}}_{n+1}} = \frac{\partial[\frac{1}{\beta h^2}\mathbf{\Phi}(\mathbf{q}_{n+1})]}{\partial \mathbf{q}_{n+1}} \cdot \frac{\partial \mathbf{q}_{n+1}}{\partial \ddot{\mathbf{q}}_{n+1}} = \frac{1}{\beta h^2}\mathbf{\Phi}_{\mathbf{q}}(\mathbf{q}_{n+1}) \cdot \beta h^2 \mathbf{I} = \mathbf{\Phi}_{\mathbf{q}}(\mathbf{q}_{n+1})$$

Solving the Nonlinear System Ψ=0
~ Newton Method ~

- Based on Newmark Integration formulas, the Jacobian is calculated as:

$$\mathbf{J} \equiv \begin{bmatrix} \frac{\partial \mathbf{\Psi}}{\partial \ddot{\mathbf{q}}} & \frac{\partial \mathbf{\Psi}}{\partial \lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{M} + \beta h^2 \left(\frac{\partial(\Phi_q^T \lambda)}{\partial \mathbf{q}} - \frac{\partial \mathbf{Q}^A}{\partial \mathbf{q}} \right) - \gamma h \frac{\partial \mathbf{Q}^A}{\partial \dot{\mathbf{q}}} & \Phi_{\mathbf{q}}^T \\ \Phi_{\mathbf{q}} & \mathbf{0} \end{bmatrix}$$

- Corrections Computed as :

$$\begin{bmatrix} \delta \ddot{\mathbf{q}} \\ \delta \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{M} + \beta h^2 \left(\frac{\partial(\mathbf{\Phi}_q^T \lambda)}{\partial \mathbf{q}} - \frac{\partial \mathbf{Q}^A}{\partial \mathbf{q}} \right) - \gamma h \frac{\partial \mathbf{Q}^A}{\partial \dot{\mathbf{q}}} & \mathbf{\Phi}_{\mathbf{q}}^T \\ \mathbf{\Phi}_{\mathbf{q}} & \mathbf{0} \end{bmatrix}^{-1} \cdot \mathbf{\Psi}(\ddot{\mathbf{q}}^{(old)}, \lambda^{old})$$

$$\begin{bmatrix} \ddot{\mathbf{q}} \\ \lambda \end{bmatrix}^{(new)} = \begin{bmatrix} \ddot{\mathbf{q}} \\ \lambda \end{bmatrix}^{(old)} - \begin{bmatrix} \delta \ddot{\mathbf{q}} \\ \delta \lambda \end{bmatrix}$$

Note: subscripts “n+1” dropped to keep presentation simple