ME451
Kinematics and Dynamics of Machine Systems

Dynamics of Planar Systems

Tuesday, November 23, 2010

Reaction Forces 6.6

Numerical Solution of IVPs

[not in the book]

Before we get started…

- Last Time
  - Example: formulating the equations of motion (EOM) for slider-crank
  - Discussed why and how to specify Initial Conditions (ICs)
- Today:
  - How the reaction forces are obtained
  - How to solve ordinary differential equations (Initial Value Problems)
- Coming up
  - Th – Thanksgiving
  - Tu, Nov. 30 – discuss numerical solution of Differential Algebraic Equations (needed to produce an approximation of the solution of the Dynamics problem)
  - Th, Dec. 2 @ 11AM in 1152ME (Dan is travelling): Hammad explains how you can add a post-processing component to simEngine2D in order to animate the motion of the mechanism (attend only if you are interested in the topic)
  - Th, Dec. 2 @ 5PM in 1152ME: Dan to hold review for midterm exam
  - Th, Dec. 2 @ 7:15-9:15 PM in 1152ME: Second midterm exam

Reaction Forces: The Framework

- Remember that we jumped through some hoops to get rid of the reaction forces that would show up in joints

- I'd like to find a way to compute them, they are important
    - Durability analysis
    - Stress/Strain analysis
    - Selecting bearings in a mechanism
    - Etc.

- It turns out that the key ingredient needed to compute the reaction forces in any joint is the set of Lagrange multipliers λ associated with that joint

Reaction Forces: The Basic Idea

- Recall the partitioning of the total force acting on our mechanical system

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_1 \\ \mathbf{Q}_2 \\ \cdots \\ \mathbf{Q}_{nb} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_1^A & + & \mathbf{Q}_1^C \\ \mathbf{Q}_2^A & + & \mathbf{Q}_2^C \\ \cdots & & \cdots \\ \mathbf{Q}_{nb}^A & + & \mathbf{Q}_{nb}^C \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_1^A \\ \mathbf{Q}_2^A \\ \cdots \\ \mathbf{Q}_{nb}^A \end{bmatrix} + \begin{bmatrix} \mathbf{Q}_1^C \\ \mathbf{Q}_2^C \\ \cdots \\ \mathbf{Q}_{nb}^C \end{bmatrix} = \mathbf{Q}^A + \mathbf{Q}^C$$

- Applying a variational approach (principle of virtual work) we ended up with this equation of motion

$$\delta \mathbf{q}^T [\mathbf{M}\ddot{\mathbf{q}} - \mathbf{Q}] = 0 \quad \Rightarrow \quad \mathbf{M}\ddot{\mathbf{q}} - \mathbf{Q} = 0 \quad \Leftrightarrow \quad \mathbf{M}\ddot{\mathbf{q}} - \mathbf{Q}^A - \mathbf{Q}^C = 0$$

- After jumping through hoops, we ended up with this:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{\Phi}_{\mathbf{q}}^T \lambda = \mathbf{Q}^A \quad \Leftrightarrow \quad \mathbf{M}\ddot{\mathbf{q}} - \mathbf{Q}^A + \mathbf{\Phi}_{\mathbf{q}}^T \lambda = \mathbf{0}$$

- It's easy to see that

$$\mathbf{Q}^C = -\mathbf{\Phi}_{\mathbf{q}}^T \lambda$$

The Important Observation

What you get when you computed λ and then premultiply by $\mathbf{\Phi}_{\mathbf{q}}^T$ is the constraint reaction force expressed as a **generalized** force:

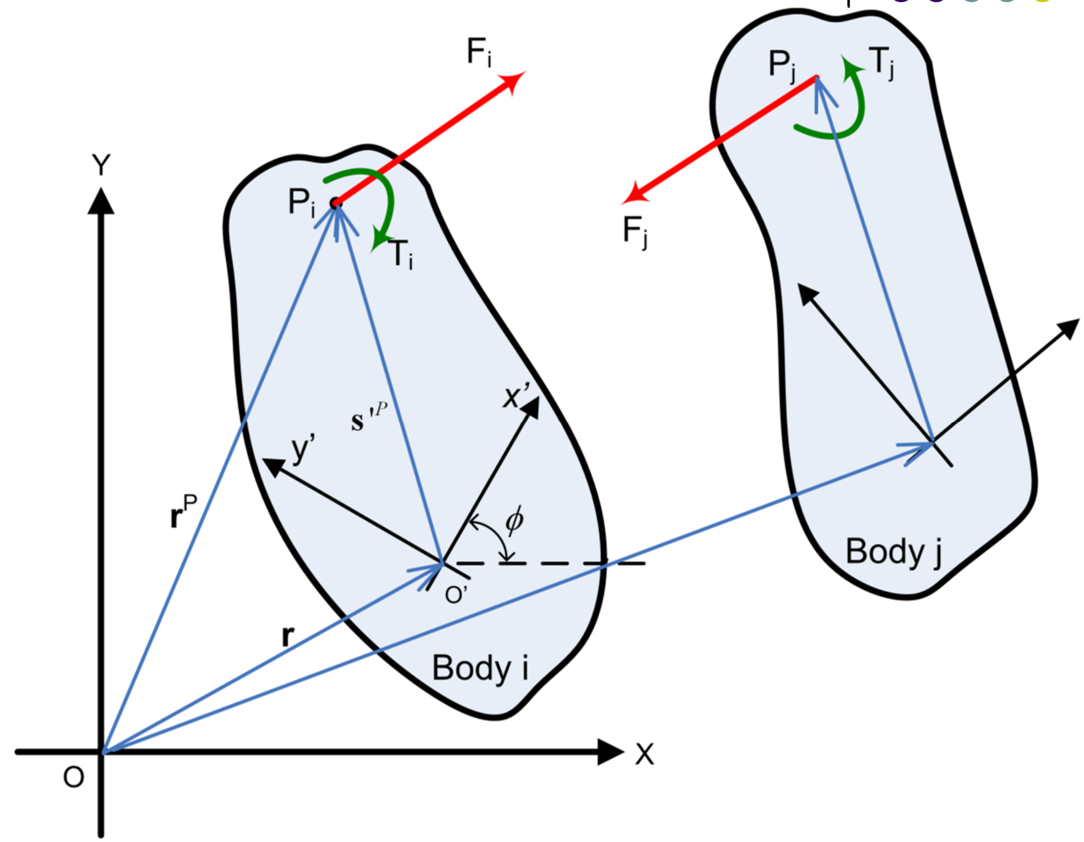
$$\mathbf{Q}^C = -\mathbf{\Phi}_{\mathbf{q}}^T \lambda$$

- IMPORTANT OBSERVATION:
  - Actually, you don't care for the "generalized" $\mathbf{Q}^C$ flavor of the reaction force, but rather you want the actual force represented in the Cartesian global reference frame
    - You'd like to have F_x, F_y, and a torque T that is due to the constraint
    - You report these quantities as they would act at a point P

- The strategy:
  - Look for a force (the classical, non-generalized flavor) and a torque, that when acting on the body at point P would lead to a generalized force equal to $\mathbf{Q}^C$

The Nuts and Bolts

- There is a joint acting between P_i and P_j and we are after finding the reaction forces/torques $\mathbf{F}_i$ and T_i, as well as $\mathbf{F}_j$ and T_j
- Figure is similar to Figure 6.6.1 out of the textbook

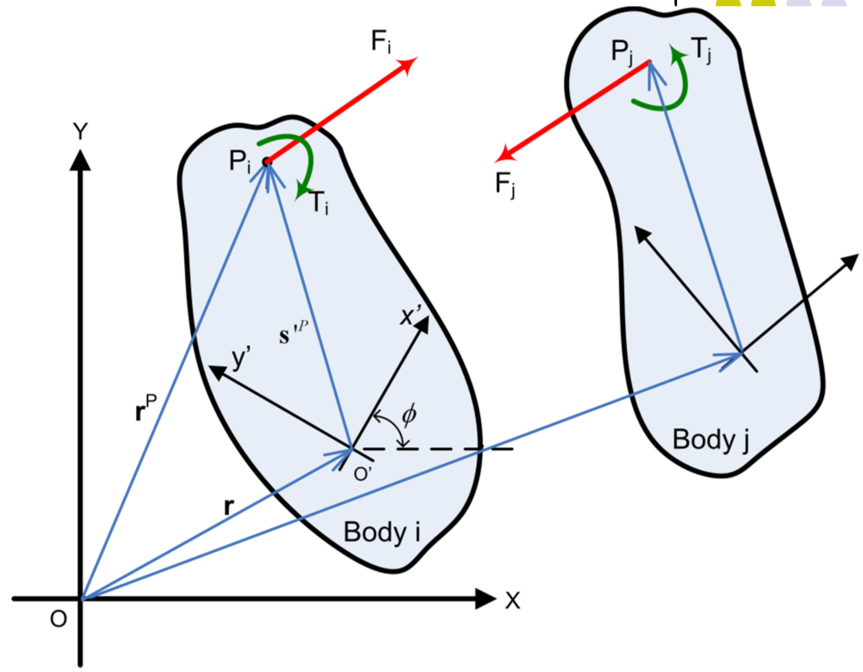


- Textbook covers topic well (pp. 234), I'm only modifying one thing:
  - The book expresses the reaction force/torque $\mathbf{F}_i$ in a body-fixed reference frame $O'x''y''$ attached at point P_i
  - I didn't see a good reason to do it that way
    - Instead, start by deriving in global reference frame OXY and then express it into the body-fixed reference frame (discussed shortly)

The Main Result
(Expression of reaction force/torque in a joint)

- Suppose that two bodies *i* and *j* are connected by a joint, and that the equation that describes that joint, which depends on the position and orientation of the two bodies, is

$$\mathbf{\Phi}(\mathbf{q}_i, \mathbf{q}_j, t) = \mathbf{\Phi}(\mathbf{r}_i, \phi_i, \mathbf{r}_j, \phi_j, t) = \mathbf{0}$$



- Suppose that the Lagrange multiplier associated with this joint is $\lambda^{(i \circ j)}$

- Then, the presence of this joint in the mechanism will lead at point P on body *i* to the presence of the following reaction force and torque:

$$\mathbf{F}_i^P = -\mathbf{\Phi}_{\mathbf{r}_i}^T \lambda^{(i \circ j)}$$

$$T = [(\mathbf{s}'^P_i)^T \mathbf{B}_i^T \mathbf{\Phi}_{\mathbf{r}_i}^T - \mathbf{\Phi}_{\phi_i}^T] \lambda^{(i \circ j)}$$

Comments
(Expression of reaction force/torque in a joint)

- Note that there is a Lagrange multiplier associated with each constraint equation
  - Number of Lagrange multipliers in mechanism is equal to number of constraints

- Each Lagrange multiplier produces (leads to) a reaction force/torque combo

- Therefore, to each constraint equation corresponds a reaction force/torque combo that throughout the time evolution of the mechanism "enforces" the satisfaction of the constraint that it is associated with
  - Example: the revolute joint brings along a set of two kinematic constraints and therefore there will be two Lagrange multipliers associated with this joint

- Since each constraint equation acts between two bodies *i* and *j*, there will also be a $\mathbf{F}_j/T_j$ combo associated with each constraint, acting on body j
  - According to Newton's third law, they oppose $\mathbf{F}_i$ and T_i, respectively

- Note that you apply the same approach when you are dealing with driving constraints (instead of kinematic constraints)
  - You will get the force and/or torque required to impose that driving constraint

Reaction Forces
~ Remember This ~

- As soon as you have a joint (constraint), you have a Lagrange multiplier λ

- As soon as you have a Lagrange multiplier you have a reaction force/torque:

The expression of $\mathbf{\Phi}^{(i \circ j)}$ for all the usual joints is known, so a boiler plate approach renders the value of the reaction force in all these joints

$$\mathbf{F}_i^P = -\mathbf{\Phi}_{\mathbf{r}_i}^T \lambda^{(i \circ j)}$$

$$T = [(\mathbf{s}'^P_i)^T \mathbf{B}_i^T \mathbf{\Phi}_{\mathbf{r}_i}^T - \mathbf{\Phi}_{\phi_i}^T] \lambda^{(i \circ j)}$$

- Just in case you want another form for the torque T above, note that

$$T = -(\mathbf{s}'^P_i)^T \mathbf{B}_i^T \mathbf{F}_i^P - \mathbf{\Phi}_{\phi_i}^T \lambda^{(i \circ j)}$$

Reporting the Reaction Force…

- There is no unique way to report ("present") the reaction force
- When you express it, it's up to you to make two selections:
  - The point P, and
  - In which RF (Reference Frame) to express it; i.e., global vs. local
- Example: represent the effect of the reaction force at the center of mass of the body and expressed it in the Global-RF

- Then,

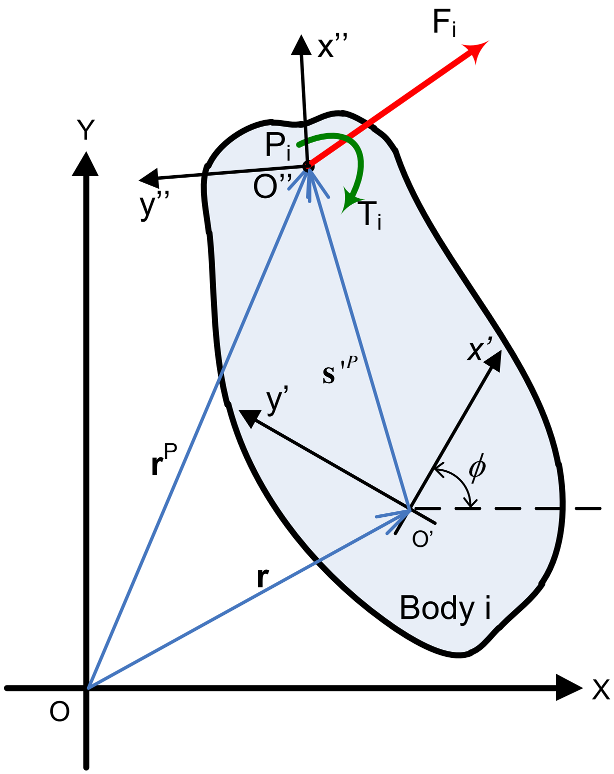
$$\bar{\mathbf{s}}_i^P = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Therefore,

$$\mathbf{F}_i^P = -\mathbf{\Phi}_{\mathbf{r}_i}^T \, \lambda^{(i \circ j)} \qquad \text{and} \qquad T = -\mathbf{\Phi}_{\phi_i}^T \, \lambda^{(i \circ j)}$$

Reporting the Reaction Force… [Cntd.]

- Assume that now you want to take the point P_i to coincide with the location of a revolute joint
- Also, you want to expressed the force in a local reference frame (L-RF) O''x''y'' attached to body "i" at this point P_i
- Basically, you represent F_i (see picture, red force) in the O''x''y'' L-RF



- Nomenclature:
  - Transformation matrix from L-RF O''x''y'' to the centroidal L-RF O'x'y': $\mathbf{C}_i$
  - Transformation matrix from centroidal L-RF to the G-RF Oxy: $\mathbf{A}_i$
- Then (this is in the book),

$$\mathbf{F}_i'' = -\mathbf{C}_i^T \ \mathbf{A}_i^T \ \mathbf{\Phi}_{\mathbf{r}_i}^T \lambda^{(i \circ j)}$$

- Note that the expression of torque doesn't change if it's represented in O''x''y'' or O'x'y' (it's a scalar)

$$T = [({\mathbf{s}'}_i^P)^T \mathbf{B}_i^T \mathbf{\Phi}_{\mathbf{r}_i}^T - \mathbf{\Phi}_{\phi_i}^T]\lambda^{(i \circ j)}$$

Example 6.6.1: Reaction force in Revolute Joint of a Simple Pendulum

Pendulum driven by motion:

$$\phi_1 = 2\pi t + \frac{3\pi}{2}$$

1) Find the reaction force in the revolute joint that connects pendulum to ground at point O
2) Express the reaction force in the $O'x_1''y_1''$ reference frame