ME451
Kinematics and Dynamics of Machine Systems

Dynamics of Planar Systems
Tuesday, November 23, 2010
Reaction Forces 6.6
Numerical Solution of IVPs
[not in the book]
Before we get started…

- **Last Time**
  - Example: formulating the equations of motion (EOM) for slider-crank
  - Discussed why and how to specify Initial Conditions (ICs)

- **Today:**
  - How the reaction forces are obtained
  - How to solve ordinary differential equations (Initial Value Problems)

- **Coming up**
  - Th – Thanksgiving
  - Tu, Nov. 30 – discuss numerical solution of Differential Algebraic Equations (needed to produce an approximation of the solution of the Dynamics problem)
  - Th, Dec. 2 @ 11AM in 1152ME (Dan is travelling): Hammad explains how you can add a post-processing component to simEngine2D in order to animate the motion of the mechanism (attend only if you are interested in the topic)
  - Th, Dec. 2 @ 5PM in 1152ME: Dan to hold review for midterm exam
  - Th, Dec. 2 @ 7:15-9:15 PM in 1152ME: Second midterm exam
Remember that we jumped through some hoops to get rid of the reaction forces that would show up in joints.

I’d like to find a way to compute them, they are important:
- Durability analysis
- Stress/Strain analysis
- Selecting bearings in a mechanism
- Etc.

It turns out that the key ingredient needed to compute the reaction forces in any joint is the set of Lagrange multipliers $\lambda$ associated with that joint.
Reaction Forces: The Basic Idea

- Recall the partitioning of the total force acting on our mechanical system

\[
Q = \begin{bmatrix}
Q_1 \\
Q_2 \\
\vdots \\
Q_{nb}
\end{bmatrix} = \begin{bmatrix}
Q_1^A + Q_1^C \\
Q_2^A + Q_2^C \\
\vdots \\
Q_{nb}^A + Q_{nb}^C
\end{bmatrix} = \begin{bmatrix}
Q_1^A \\
Q_2^A \\
\vdots \\
Q_{nb}^A
\end{bmatrix} + \begin{bmatrix}
Q_1^C \\
Q_2^C \\
\vdots \\
Q_{nb}^C
\end{bmatrix} = Q^A + Q^C
\]

- Applying a variational approach (principle of virtual work) we ended up with this equation of motion

\[
\delta q^T [M\ddot{q} - Q] = 0 \quad \Rightarrow \quad M\ddot{q} - Q = 0 \quad \Leftrightarrow \quad M\ddot{q} - Q^A - Q^C = 0
\]

- After jumping through hoops, we ended up with this:

\[
M\ddot{q} + \Phi_q^T \lambda = Q^A \quad \Leftrightarrow \quad M\ddot{q} - Q^A + \Phi_q^T \lambda = 0
\]

- It’s easy to see that

\[
Q^C = -\Phi_q^T \lambda
\]
**The Important Observation**

What you get when you computed $\lambda$ and then premultiply by $\Phi_q^T$ is the constraint reaction force expressed as a **generalized** force:

$$Q^C = -\Phi_q^T \lambda$$

- **IMPORTANT OBSERVATION:**
  - Actually, you don’t care for the “generalized” $Q^C$ flavor of the reaction force, but rather you want the actual force represented in the Cartesian global reference frame:
    - You’d like to have $F_x$, $F_y$, and a torque $T$ that is due to the constraint
    - You report these quantities as they would act at a point $P$

- The strategy:
  - Look for a force (the classical, non-generalized flavor) and a torque, that when acting on the body at point $P$ would lead to a generalized force equal to $Q^C$
The Nuts and Bolts

- There is a joint acting between \( P_i \) and \( P_j \) and we are after finding the reaction forces/torques \( F_i \) and \( T_i \), as well as \( F_j \) and \( T_j \)
- Figure is similar to Figure 6.6.1 out of the textbook

- Textbook covers topic well (pp. 234), I’m only modifying one thing:
  - The book expresses the reaction force/torque \( F_i \) in a body-fixed reference frame \( O'x'y'z' \) attached at point \( P_i \)
  - I didn’t see a good reason to do it that way
  - Instead, start by deriving in global reference frame \( OXY \) and then express it into the body-fixed reference frame (discussed shortly)
The Main Result
(Expression of reaction force/torque in a joint)

- Suppose that two bodies $i$ and $j$ are connected by a joint, and that the equation that describes that joint, which depends on the position and orientation of the two bodies, is

$$\Phi(q_i, q_j, t) = \Phi(r_i, \phi_i, r_j, \phi_j, t) = 0$$

- Suppose that the Lagrange multiplier associated with this joint is $\lambda^{(i\circ j)}$

- Then, the presence of this joint in the mechanism will lead at point P on body $i$ to the presence of the following reaction force and torque:

$$F_i^P = -\Phi^T r_i \lambda^{(i\circ j)}$$

$$T = [(s_i^P)^T B_i^T \Phi^T r_i - \Phi^T \phi_i] \lambda^{(i\circ j)}$$
Comments
(Express the reaction force/torque in a joint)

- Note that there is a Lagrange multiplier associated with each constraint equation
  - Number of Lagrange multipliers in mechanism is equal to number of constraints

- Each Lagrange multiplier produces (leads to) a reaction force/torque combo

- Therefore, to each constraint equation corresponds a reaction force/torque combo that throughout the time evolution of the mechanism “enforces” the satisfaction of the constraint that it is associated with
  - Example: the revolute joint brings along a set of two kinematic constraints and therefore there will be two Lagrange multipliers associated with this joint

- Since each constraint equation acts between two bodies \( i \) and \( j \), there will also be a \( F_j/T_j \) combo associated with each constraint, acting on body \( j \)
  - According to Newton’s third law, they oppose \( F_i \) and \( T_i \), respectively

- Note that you apply the same approach when you are dealing with driving constraints (instead of kinematic constraints)
  - You will get the force and/or torque required to impose that driving constraint
Reaction Forces
~ Remember This ~

- As soon as you have a joint (constraint), you have a Lagrange multiplier $\lambda$

- As soon as you have a Lagrange multiplier you have a reaction force/torque:

$$ F_i^P = -\Phi^T_{r_i} \lambda^{(i \circ j)} $$

$$ T = [(s_i^P)^T B_i^T \Phi^T_{r_i} - \Phi^T_{\phi_i}] \lambda^{(i \circ j)} $$

- Just in case you want another form for the torque $T$ above, note that

$$ T = -(s_i^P)^T B_i^T F_i^P - \Phi^T_{\phi_i} \lambda^{(i \circ j)} $$

The expression of $\Phi^{(i \circ j)}$ for all the usual joints is known, so a boiler plate approach renders the value of the reaction force in all these joints.
Reporting the Reaction Force…

- There is no unique way to report ("present") the reaction force.
- When you express it, it’s up to you to make two selections:
  - The point P, and
  - In which RF (Reference Frame) to express it; i.e., global vs. local.
- Example: represent the effect of the reaction force at the center of mass of the body and expressed it in the Global-RF.

  - Then,
    \[
    \tilde{s}_i^P = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
    \]

  - Therefore,
    
    \[
    F_i^P = -\Phi^T_{r_i} \lambda^{(i\circ j)} \quad \text{and} \quad T = -\Phi^T_{\phi_i} \lambda^{(i\circ j)}
    \]
Assume that now you want to take the point $P_i$ to coincide with the location of a revolute joint.

Also, you want to express the force in a local reference frame (L-RF) $O''x''y''$ attached to body “$i$” at this point $P_i$.

Basically, you represent $F_i$ (see picture, red force) in the $O''x''y''$ L-RF.

Nomenclature:
- Transformation matrix from L-RF $O''x''y''$ to the centroidal L-RF $O'x'y'$: $C_i$
- Transformation matrix from centroidal L-RF to the G-RF $Oxy$: $A_i$

Then (this is in the book),
$$F''_i = -C_i^T A_i^T \Phi_{r_i}^T \lambda^{(i\circ j)}$$

Note that the expression of torque doesn’t change if it’s represented in $O''x''y''$ or $O'x'y'$ (it’s a scalar)
$$T = [(s_i^P)^T B_i^T \Phi_{r_i}^T - \Phi_{\phi_i}^T] \lambda^{(i\circ j)}$$
Example 6.6.1: Reaction force in Revolute Joint of a Simple Pendulum

Pendulum driven by motion:
\[ \phi_1 = 2\pi t + \frac{3\pi}{2} \]

1) Find the reaction force in the revolute joint that connects pendulum to ground at point O

2) Express the reaction force in the \( O'x''y'' \) reference frame