ME451
Kinematics and Dynamics of Machine Systems

Dynamics of Planar Systems

November 11, 2010

6.1.3, 6.1.4, 6.2, starting 6.3

Before we get started…

- Last Time
  - Wrapped up the derivation of the EOM for planar rigid bodies

- Today
  - Example
  - Look into inertia properties of 2D geometries
    - Center of mass
    - Parallel axis theorem
    - Mass moment of inertia for composite geometries
  - Discuss the concept of generalized force
  - Look into two types of concentrated forces/torques: TSDA/RSDA

- Assignment
  - HW (on Nov 8): Problem 6.2.1
  - MATLAB assignment: Kinematics analysis of simple mechanism
  - ADAMS: emailed by the TA

[Before doing the example, recall this from last lecture]

On the meaning of $[\mathbf{B}\bar{\mathbf{s}}^P]^T \cdot \mathbf{F}^P$

- My claim is that $[\mathbf{B}\bar{\mathbf{s}}^P]^T \cdot \mathbf{F}^P$ represents the torque of $\mathbf{F}^P$ about the origin of the L-RF. In what follows we'll confirm that this is true.

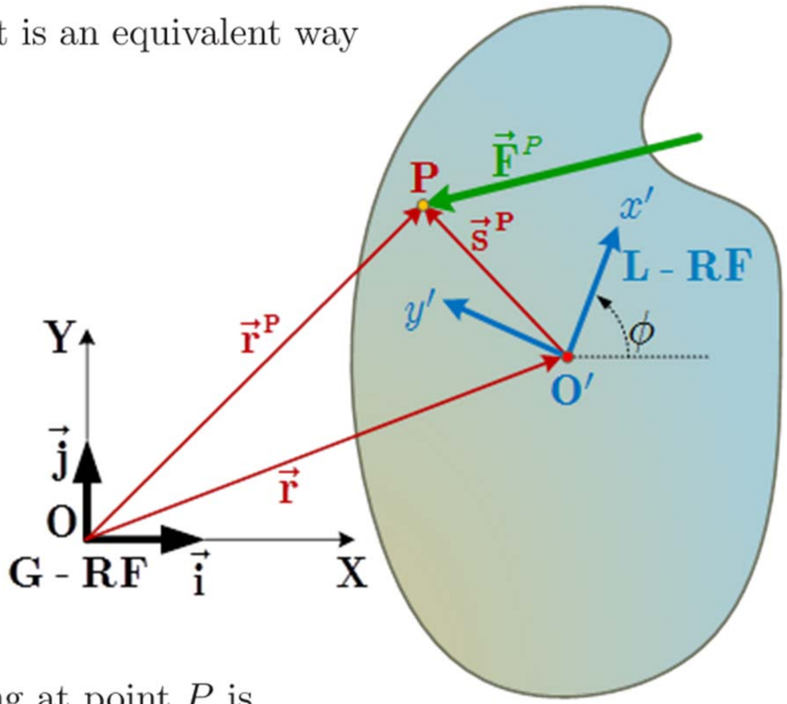
- The size of the torque $\mathbf{r}^P \times \mathbf{F}^P$ is $||\mathbf{s}^P|| \cdot ||\mathbf{F}^P|| \cdot \sin\theta$. What is an equivalent way to express this quantity?

- Use the identity

$$\sin\theta = \cos(\theta - \frac{\pi}{2})$$

- Then, the value of the torque, is

$$\begin{aligned} n^P &= ||\mathbf{s}^P|| \cdot ||\mathbf{F}^P|| \sin\theta \\ &= ||\mathbf{s}^P|| \cdot ||\mathbf{F}^P|| \cos(\theta - \tfrac{\pi}{2}) \\ &= [\mathbf{s}^{P\perp}]^T \cdot \mathbf{F}^P \\ &= [\mathbf{B}\bar{\mathbf{s}}^P]^T \cdot \mathbf{F}^P \end{aligned}$$



- To conclude, the value of the torque produced by $\mathbf{F}^P$ acting at point P is

$$n^P = [\mathbf{B}\bar{\mathbf{s}}^P]^T \cdot \mathbf{F}^P$$

- A concentrated (point) force leads to the presence of a torque that affects the rotation of the body (the rotational degree of freedom associated with the motion of the body).

Example 6.1.1

- Tractor model: derive equations of motion

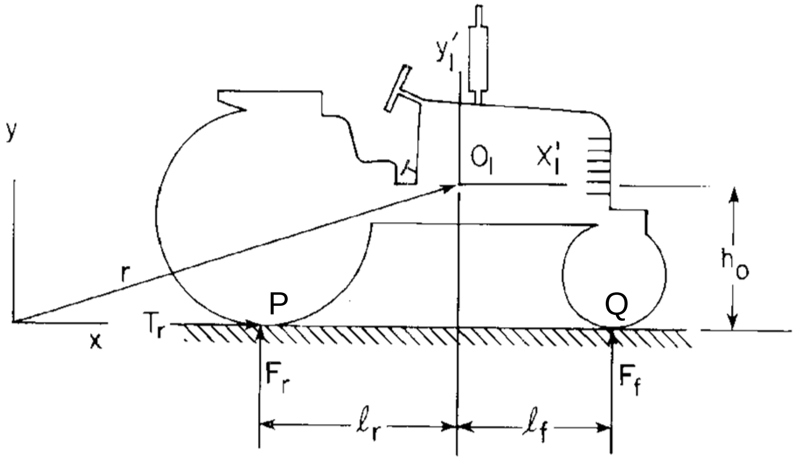


Figure 6.1.2 Plane motion of a tractor.

- Traction force T_r
- Small angle assumption (pitch angle ϕ)
- Force in tires depends on tire deflection:

$$f(d) = \begin{cases} kd, & d \geq 0 \\ 0, & d < 0 \end{cases}$$

End: 6.1.3

Begin: 6.1.4

Inertia Properties of Rigid Bodies

- Issues discussed:
  - Determining location of the center of mass (centroid) of a rigid body
  - Parallel axis theorem
  - Mass moment of inertia of composite bodies
  - Location of centroid for a rigid body with a symmetry axis

Location of the Center of Mass

- By <u>definition</u> the centroid is that point of a body for which, when you place a reference frame at that point and compute a certain integral in relation to that reference frame, the integral vanishes:

$$\int_m \mathbf{s}'^P \, dm(P) = 0$$

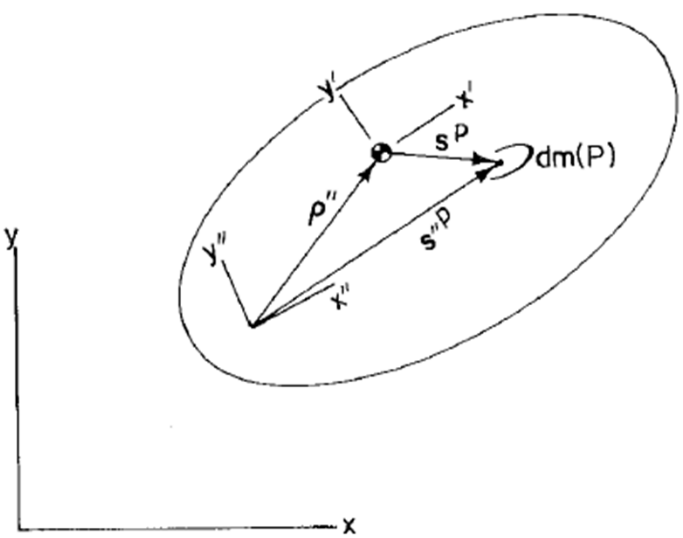


Figure 6.1.3 Location of a centroid.

Q: How can I determine the location of the center of mass?

Mass Moment of Inertia (MMI)

- The MMI was defined as:

$$J' = \int_m \left[\mathbf{s}'^P\right]^T \mathbf{s}'^P \, d\mathrm{m}(P)$$

- Suppose that the reference frame with respect to which the integral is computed is a centroidal RF

- You might ask yourself, what happens if I decide to evaluate the integral above with respect to a different reference frame, located at a different point attached to this body?

- Answer is provided by the parallel axis theorem (ρ is vector from centroidal reference frame to the new reference frame) :

$$\boxed{J'_{new} = J' + m\left(\rho^T \rho\right)}$$

MMI of a Composite Body

- Step 1: Compute the centroid of the composite body

$$\rho'' = \frac{1}{m} \sum_{i=1}^{k} m_i \rho_i''$$

- Step 2: For each sub-body, apply the parallel axis theorem to compute the MMI of that sub-body with respect to the newly computed centroid

$$J^* = \sum_{i=1}^{k} \left(J_i' + m_i \rho_i^{*T} \rho_i^* \right)$$

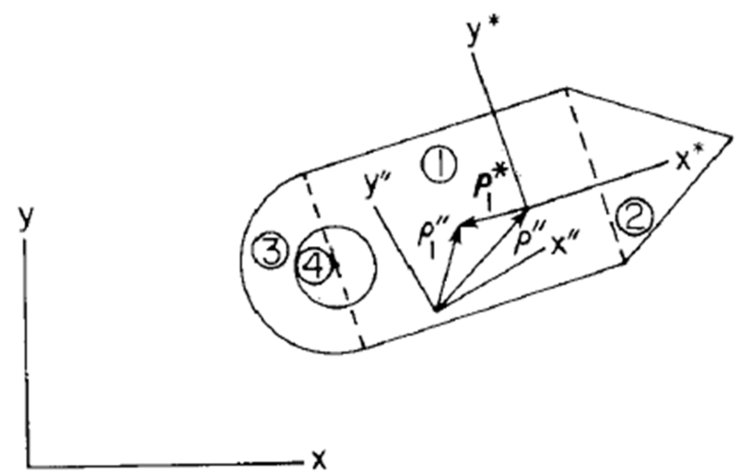


Figure 6.1.5 Body made up of subcomponents.

- Note: if holes are present in the composite body, it's ok to add and subtract material (this translates into positive and negative mass)

Location of the Center of Mass (Cntd.)

- What can one say if the rigid body has a symmetry axis?
  - Here symmetry axis means that **both** mass and geometry are symmetric with respect to that axis

- You can say this: the centroid is somewhere along that axis

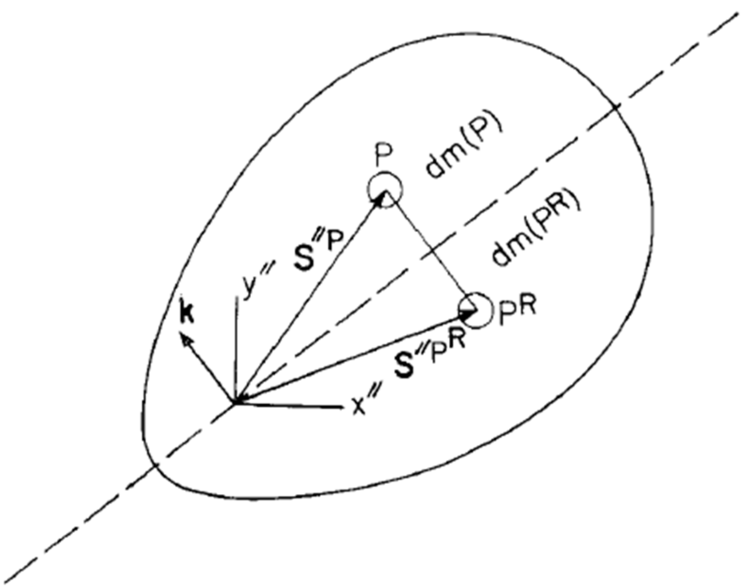


Figure 6.1.4 Body with axis of symmetry.

- NOTE: if the rigid body has two axes of symmetry, the centroid is on each of them, and therefore is where they intersect

Notation & Nomenclature

- We'll formulate the problem in a concise fashion using matrix-vector notation:

$$\delta \mathbf{r}^T [m\ddot{\mathbf{r}} - \mathbf{F}] + \delta\phi[J'\ddot{\phi} - n] = 0$$

$$\Downarrow$$

$$\delta \mathbf{q}^T [\mathbf{M}\ddot{\mathbf{q}} - \mathbf{Q}] = 0$$

$$\mathbf{q} = \begin{bmatrix} \mathbf{r} \\ \phi \end{bmatrix} \qquad \delta\mathbf{q} = \begin{bmatrix} \delta\mathbf{r} \\ \delta\phi \end{bmatrix} \qquad \mathbf{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J' \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} \mathbf{F} \\ n \end{bmatrix}$$

So what do I actually mean when I talk about "generalized forces"?

~ I mean the **Q** above ~

Generalized Forces: Problem Context

- **Q**: How are **F** and *n* in the equations of motion obtained (specified)?

$$\begin{aligned} m\ddot{\mathbf{r}} - \mathbf{F} &= \mathbf{0} \qquad \text{Equations of Motion governing translation} \\ J'\ddot{\phi} - n &= 0 \qquad \text{Equation of Motion governing rotation} \end{aligned}$$

- After all, where are the original **F** and *n* coming from?

$m\,\mathbf{a} = \mathbf{f}$ → Integral manipulations that rely on the rigid-body assumption → $\delta\mathbf{r}^T[m\ddot{\mathbf{r}} - \mathbf{F}] + \delta\phi[J'\ddot{\phi} - n] = 0$

Newton's second law for a **particle**

Holds for the **entire** body. It says that total virtual work should stay zero (in other words, when some forces kick in, the accelerations should change such that the overall virtual work remains zero)

This holds the key to how a force gets into the picture. We are not that much interested in the expression of the force per se, but rather in the amount of virtual work that it brings into the picture, because virtual work is what's important…

Generalized Forces: How to Deal with Them

- **F** was the sum of all distributed forces **f**(P) acting per unit mass:

$$\mathbf{F} = \int_m \mathbf{f}(P)\,\mathrm{d}m(P) \qquad \text{(Eq. 6.1.16)}$$

- **n** was the torque produced by the forces **f**(P)

$$n = \int_m \left(\mathbf{B}\bar{\mathbf{s}}^P\right)^T \mathbf{f}(P)\,\mathrm{d}m(P) = \int_m \left(\bar{\mathbf{s}}^{P^\perp}\right)^T \bar{\mathbf{f}}(P)\,\mathrm{d}m(P) \qquad \text{(Eq. 6.1.17)}$$

- <u>QUESTION</u>: What happens when we don't have distributed forces, such as **f**(P), at each point P on the body, but rather a force acting at only *one* point P of the body?