ME451
Kinematics and Dynamics of Machine Systems

Dynamics of Planar Systems

November 09, 2010

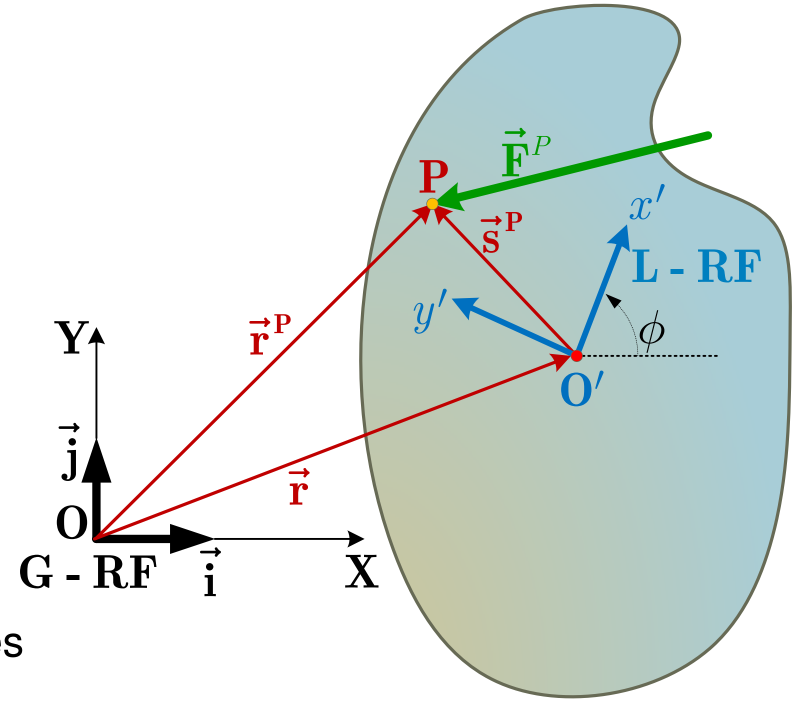
6.1.3, 6.1.4

Before we get started…

- Last Time
  - Started the derivation of the EOM for one 2D rigid body in planar motion
    - We started from “scratch”
    - The assumptions used:
      - Body was rigid
      - Working with centroidal body reference frame. Simplifies form of the equations of motion
- Today
  - Finish the derivation (for one body)
  - Look at an example
  - Analyze inertia properties of rigid bodies
- Due Th: take-home exam + older MATLAB assignment
- Coming up:
  - Final Projects topic should be clear by Th (see me if you are not clear or want to change)
  - Next exam
    - On Dec 2 (Th) in the *evening*: 7:15 – 9:15 PM
    - Review session: on Dec. 2, 5-7 PM
    - No regular lecture on Th, Dec. 2

Types of Forces & Torques Acting on a Body

- Type 1: Distributed over the volume of a body
  - Inertia forces
  - Distributed forces
  - Internal interaction forces



- Type 2: Concentrated at a point
  - Action (or applied, or external) forces and torques
  - Reaction (or constraint) forces and torques

Virtual Work: Dealing with Inertia Forces

- Framework: we are considering a point P of body i. This point is associated with an infinitesimal mass element $dm_i(P)$
- Expression of the force:

$$-\ddot{\mathbf{r}}_i^P \; dm_i(P)$$

- Virtual work produced:

$$[\delta \mathbf{r}_i^P]^T \cdot [-\ddot{\mathbf{r}}_i^P \; dm_i(P)]$$

- Comments:
  - The total virtual work produced by this type of force is obtained by summing over all points of body i:

$$\int_{m_i} -[\delta \mathbf{r}_i^P]^T \cdot \ddot{\mathbf{r}}_i^P \; dm_i(P)$$

Virtual Work: Dealing with Mass-Distributed Forces

- Framework: we are considering a point P of body i. This point is associated with an infinitesimal mass element $dm_i(P)$. A force per unit mass, $\mathbf{f}_i(P)$, is assumed to act at point P.
- Expression of the force:

$$\mathbf{f}_i(P)\, dm_i(P)$$

- Virtual work produced:

$$[\delta \mathbf{r}_i^P]^T \cdot \mathbf{f}_i(P)\, dm_i(P)$$

- Comments:
  - The total virtual work produced by this type of force is obtained by summing over all points of body i:

$$\int_{m_i} [\delta \mathbf{r}_i^P]^T \cdot \mathbf{f}_i(P)\, dm_i(P)$$

  - This type of force is rarely seen in classical multibody dynamics. Exception: the force due to the gravitational field, which leads to the weight of the body. In this case $\mathbf{f}_i(P) = \mathbf{g}$, where $\mathbf{g}$ is the gravitational acceleration of magnitude $g \approx 9.81 \frac{m}{s^2}$ (in Madison, WI).

Virtual Work: Dealing with Internal Interaction Forces

- Framework: we are considering a point P of body i. This point is associated with an infinitesimal mass element $dm_i(P)$. We also consider an *arbitrary* point R on body i. The focus is on the **internal force** acting between the mass elements $dm_i(P)$ and $dm_i(R)$.

- The expression of this type of force acting at point P is obtained by considering the contribution of each point R of the body:

$$\int_{m_i} \mathbf{f}_i(P,R)\, dm_i(R)$$

- Virtual work produced:

$$[\delta \mathbf{r}_i^P]^T \cdot \int_{m_i} \mathbf{f}_i(P,R)\, dm_i(R)$$

- Comments:

  – The total virtual work produced by this type of force when acting at all points of body i:

$$\int_{m_i} [\delta \mathbf{r}_i^P]^T \left[\int_{m_i} \mathbf{f}_i(P,R)\, dm_i(R) \right] dm_i(P) = \int_{m_i} \int_{m_i} [\delta \mathbf{r}_i^P]^T \cdot \mathbf{f}_i(P,R)\, dm_i(R)\, dm_i(P)$$

  – The assumption that we make is that the force $\mathbf{f}_i(P,R)$ acts along the line connecting points P and R. In other words, $\mathbf{f}_i(P,R) dm_i(R) = k(\mathbf{r}_i^P - \mathbf{r}_i^R)$, where k is a scalar that might depend on the points P and R.

Dealing with Active Forces

- Framework: We assume that a set of active forces acts on body i. These active forces are acting at a collection of points generically denoted by $\mathcal{U}_i$.
- Expression of this type of force acting at point $U \in \mathcal{U}_i$:

$$\mathbf{F}_U^a$$

- Virtual work produced by this set of forces:

$$\sum_{U \in \mathcal{U}_i} [\delta \mathbf{r}_i^U]^T \cdot \mathbf{F}_U^a$$

Dealing with Active Torques

- Framework: We assume that a set of active torques acts on body i. We will assume that these active torques are acting at a collection of points generically denoted by $\mathcal{V}_i$.
- Expression of this type of torque acting at point $V \in \mathcal{V}_i$, expressed in the L-RF$_i$:

$$\bar{\mathbf{n}}_V^a$$

- Virtual work produced by this set of torques:

$$\sum_{V \in \mathcal{V}_i} [\delta\phi_i]^T \cdot \bar{\mathbf{n}}_V^a$$

- Comments: Note that since we are talking about *rigid* bodies, we have the same virtual rotation $\delta\phi_i$ no matter which of the torques acting on the rigid body we are dealing with

Virtual Work: Dealing with Constraint Reaction Forces

- Framework: We assume that a set of constraints acts on body i. These constraints most often lead to the presence of reaction forces. We will assume that the constraints on body i are producing reaction (constraint) forces acting at a collection of points generically denoted by $\mathcal{Q}_i$.
- Expression of this type of force acting at point $Q \in \mathcal{Q}_i$:

$$\mathbf{F}_Q^r$$

- Virtual work produced by this set of forces:

$$\sum_{Q \in \mathcal{Q}_i} [\delta \mathbf{r}_i^Q]^T \cdot \mathbf{F}_Q^r$$

- Comments:
  - One of the outcomes of solving the EOM will be to compute the value of the reaction forces $\mathbf{F}_Q^r$ for $Q \in \mathcal{Q}_i$
  - A separate discussion will follow on the meaning of the points $\mathcal{Q}_i$

Virtual Work: Dealing with Constraint Reaction Torques

- Framework: We assume that a set of constraints acts on body i. These constraints might lead to the presence of reaction torques. We will assume that the constraints on body i are producing reaction (constraint) torques acting at a collection of points generically denoted by $\mathcal{Z}_i$.
- Expression of this type of torque acting at point $Z \in \mathcal{Z}_i$ when represented in the L-RF (superscript r stands for reaction):

$$\bar{\mathbf{n}}_Z^r$$

- Virtual work produced by these reaction torques:

$$\sum_{Z \in \mathcal{Z}_i} [\delta \phi_i]^T \cdot \bar{\mathbf{n}}_Z^r$$

- Comments:
  - One of the outcomes of solving the EOM will be to compute the value of the reaction torques $\bar{\mathbf{n}}_Z^r$ for $Z \in \mathcal{Z}_i$

Deriving Newton's Equations for a <u>body</u> with planar motion

- NOTE:
  - You should be able to derive Newton's equations for a planar rigid body on your own (closed books)
  - Overall, the book does a <u>very good job</u> in explaining the derivation the equations of motion (EOM) for a rigid body
  - No notes last time, the material is straight out of the book (page 200)

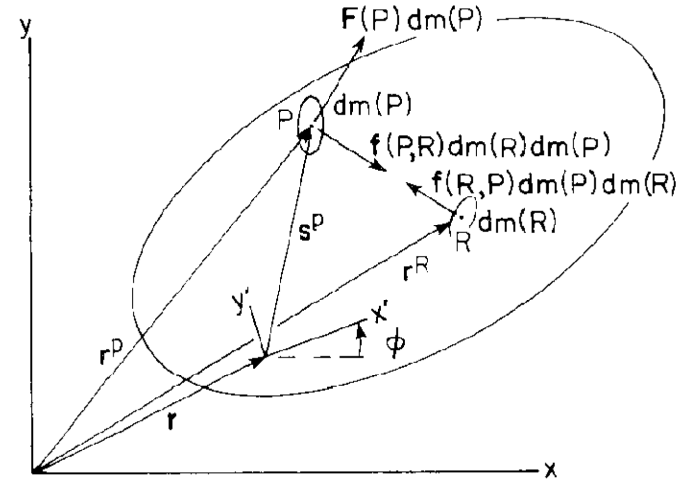


Figure 6.1.1 Forces acting on a rigid body.

EOM: First Pass

- For now, assume that there are no concentrated forces

- Do this for *one* body for now

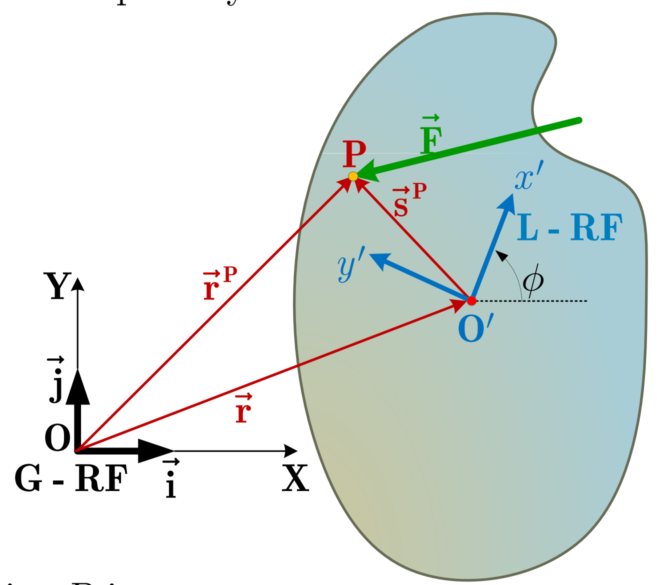
- We are going to deal with the distributed forces and use them in the context of d'Alembert's Principle
  - Inertia forces
  - Internal forces
  - Other distributed forces (gravity)

On the meaning of $[\mathbf{B}\bar{\mathbf{s}}^P]^T \cdot \mathbf{F}$

- In order to get the torque of $\mathbf{F}$ about the origin of the L-RF, I need to take the cross product of $\mathbf{s}^P \times \mathbf{F}$. I'm interested in the value of this torque.
- The value is $||\mathbf{s}^P||\cdot||\mathbf{F}||\cdot\sin\theta$. What is an equivalent way to express this quantity?
- Use the identity

$$\sin\theta = \cos(\theta - \frac{\pi}{2})$$

- Then, the value of the torque, is



$$\begin{aligned} n &= ||\mathbf{s}^P||\cdot||\mathbf{F}||\sin\theta \\ &= ||\mathbf{s}^P||\cdot||\mathbf{F}||\cos(\theta - \tfrac{\pi}{2}) \\ &= [\mathbf{s}^{P\perp}]^T\cdot\mathbf{F} \\ &= [\mathbf{B}\bar{\mathbf{s}}^P]^T\cdot\mathbf{F} \end{aligned}$$

- To conclude, the value of the torque produced by $\mathbf{F}$ acting at point P is

$$n = [\mathbf{B}\bar{\mathbf{s}}^P]^T\cdot\mathbf{F}$$

Example 6.1.1

- Tractor model: derive equations of motion

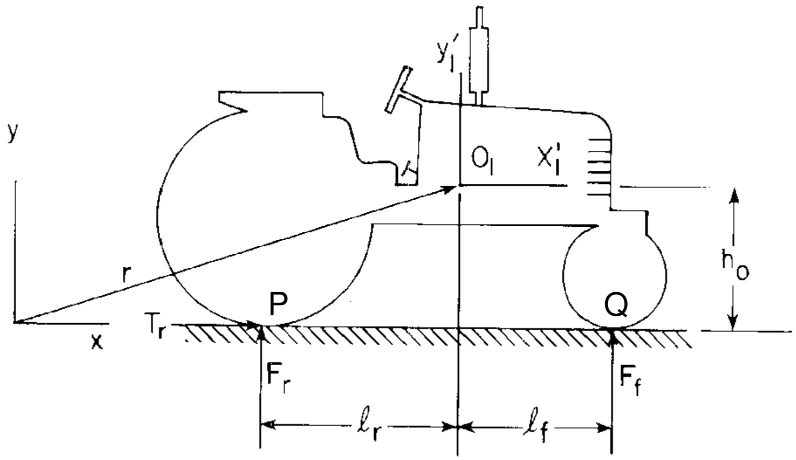


Figure 6.1.2 Plane motion of a tractor.

- Traction force T_r
- Small angle assumption (pitch angle ϕ)
- Force in tires depends on tire deflection:

$$f(d) = \begin{cases} kd, & d \geq 0 \\ 0, & d < 0 \end{cases}$$

End: 6.1.3
Begin: 6.1.4

Inertia Properties of Rigid Bodies

- Issues discussed:
  - Determining location of the center of mass (centroid) of a rigid body
  - Parallel axis theorem
  - Mass moment of inertia of composite bodies
  - Location of centroid for a rigid body with a symmetry axis

Location of the Center of Mass

- By definition the centroid is that point of a body for which, when you place a reference frame at that point and compute a certain integral in relation to that reference frame, the integral vanishes:

$$\int_m \mathbf{s}'^P \, dm(P) = 0$$

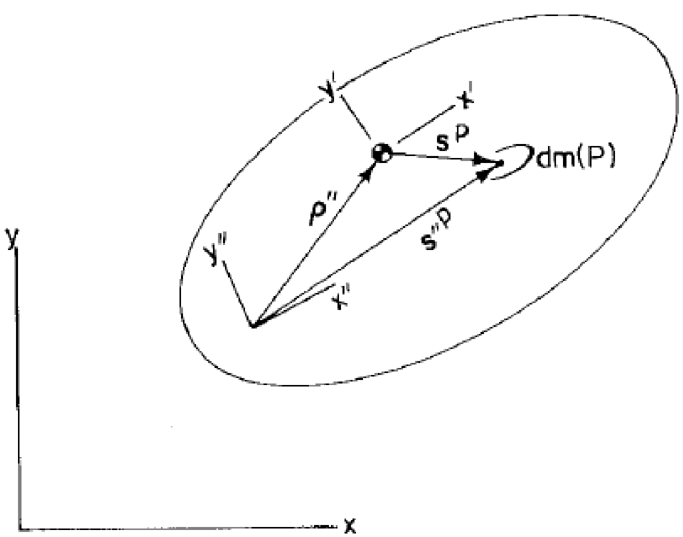


Figure 6.1.3 Location of a centroid.

Q: How can I determine the location of the center of mass?

Mass Moment of Inertia (MMI)

- The MMI was defined as:

$$J' = \int_m \left[\mathbf{s}'^P\right]^T \mathbf{s}'^P \, d\mathrm{m}(P)$$

- Suppose that the reference frame with respect to which the integral is computed is a centroidal RF

- You might ask yourself, what happens if I decide to evaluate the integral above with respect to a different reference frame, located at a different point attached to this body?

- Answer is provided by the parallel axis theorem (ρ is vector from centroidal reference frame to the new reference frame) :

$$\boxed{J'_{new} = J' + m\,(\rho^T \rho)}$$

MMI of a Composite Body

- Step 1: Compute the centroid of the composite body

$$\rho'' = \frac{1}{m} \sum_{i=1}^{k} m_i \rho_i''$$

- Step 2: For each sub-body, apply the parallel axis theorem to compute the MMI of that sub-body with respect to the newly computed centroid

$$J^* = \sum_{i=1}^{k} (J_i' + m_i \rho_i^{*T} \rho_i^*)$$

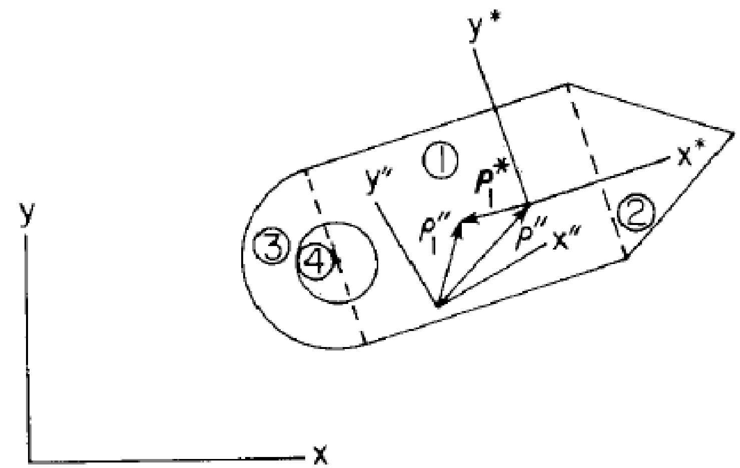


Figure 6.1.5 Body made up of subcomponents.

- Note: if holes are present in the composite body, it's ok to add and subtract material (this translates into positive and negative mass)

Location of the Center of Mass (Cntd.)

- What can one say if the rigid body has a symmetry axis?
  - Here symmetry axis means that **both** mass and geometry are symmetric with respect to that axis

- You can say this: the centroid is somewhere along that axis

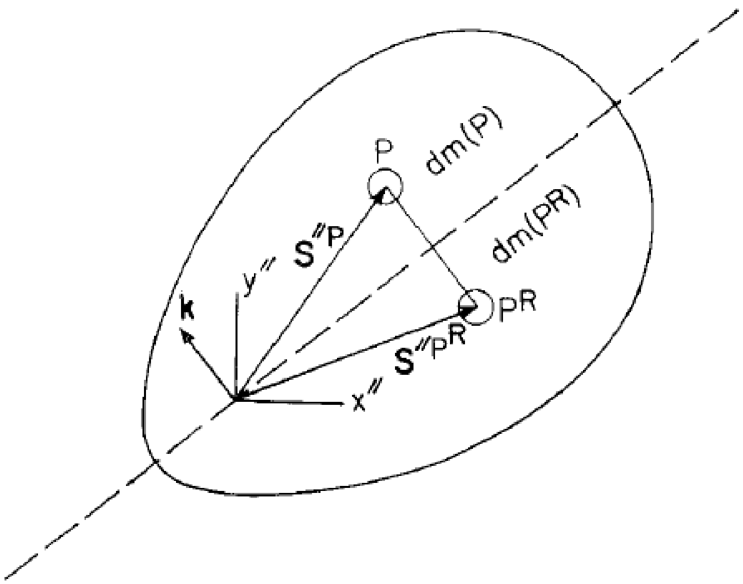


Figure 6.1.4 Body with axis of symmetry.

- NOTE: if the rigid body has two axes of symmetry, the centroid is on each of them, and therefore is where they intersect

Determining the Expression of Generalized Forces (6.2)