ME451
Kinematics and Dynamics of Machine Systems

Dynamics of Planar Systems
November 09, 2010
6.1.3, 6.1.4

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ME451, UW-Madison
Before we get started...

- Last Time
  - Started the derivation of the EOM for one 2D rigid body in planar motion
    - We started from “scratch”
    - The assumptions used:
      - Body was rigid
      - Working with centroidal body reference frame. Simplifies form of the equations of motion

- Today
  - Finish the derivation (for one body)
  - Look at an example
  - Analyze inertia properties of rigid bodies

- Due Th: take-home exam + older MATLAB assignment

- Coming up:
  - Final Projects topic should be clear by Th (see me if you are not clear or want to change)
  - Next exam
    - On Dec 2 (Th) in the *evening*: 7:15 – 9:15 PM
    - Review session: on Dec. 2, 5-7 PM
    - No regular lecture on Th, Dec. 2
Types of Forces & Torques Acting on a Body

- **Type 1:** Distributed over the volume of a body
  - Inertia forces
  - Distributed forces
  - Internal interaction forces

- **Type 2:** Concentrated at a point
  - Action (or applied, or external) forces and torques
  - Reaction (or constraint) forces and torques
Virtual Work: Dealing with Inertia Forces

- Framework: we are considering a point $P$ of body $i$. This point is associated with an infinitesimal mass element $dm_i(P)$

- Expression of the force:
  $$-\ddot{r}_i^P dm_i(P)$$

- Virtual work produced:
  $$[\delta r_i^P]^T \cdot [-\ddot{r}_i^P dm_i(P)]$$

- Comments:
  - The total virtual work produced by this type of force is obtained by summing over all points of body $i$:
  $$\int_{m_i} -[\delta r_i^P]^T \cdot \ddot{r}_i^P dm_i(P)$$
Virtual Work: Dealing with Mass-Distributed Forces

- Framework: we are considering a point $P$ of body $i$. This point is associated with an infinitesimal mass element $dm_i(P)$. A force per unit mass, $f_i(P)$, is assumed to act at point $P$.

- Expression of the force:
  
  \[ f_i(P) \, dm_i(P) \]

- Virtual work produced:
  
  \[ [\delta r_i^P]^T \cdot f_i(P) \, dm_i(P) \]

- Comments:
  
  - The total virtual work produced by this type of force is obtained by summing over all points of body $i$:
    
    \[ \int_{m_i} [\delta r_i^P]^T \cdot f_i(P) \, dm_i(P) \]

  - This type of force is rarely seen in classical multibody dynamics. Exception: the force due to the gravitational field, which leads to the weight of the body. In this case $f_i(P) = g$, where $g$ is the gravitational acceleration of magnitude $g \approx 9.81 \frac{m}{s^2}$ (in Madison, WI).
Virtual Work: Dealing with Internal Interaction Forces

- Framework: we are considering a point $P$ of body $i$. This point is associated with an infinitesimal mass element $dm_i(P)$. We also consider an arbitrary point $R$ on body $i$. The focus is on the internal force acting between the mass elements $dm_i(P)$ and $dm_i(R)$.

- The expression of this type of force acting at point $P$ is obtained by considering the contribution of each point $R$ of the body:

$$\int_{m_i} f_i(P, R) \, dm_i(R)$$

- Virtual work produced:

$$[\delta \mathbf{r}_i^P]^T \cdot \int_{m_i} f_i(P, R) \, dm_i(R)$$

- Comments:
  - The total virtual work produced by this type of force when acting at all points of body $i$:

$$\int_{m_i} \left[ [\delta \mathbf{r}_i^P]^T \cdot \int_{m_i} f_i(P, R) \, dm_i(R) \right] \, dm_i(P) = \int \int \left[ [\delta \mathbf{r}_i^P]^T \cdot f_i(P, R) \right] \, dm_i(R) \, dm_i(P)$$

  - The assumption that we make is that the force $f_i(P, R)$ acts along the line connecting points $P$ and $R$. In other words, $f_i(P, R)dm_i(R) = k(r_i^P - r_i^R)$, where $k$ is a scalar that might depend on the points $P$ and $R$. 

Dealing with Active Forces

- Framework: We assume that a set of active forces acts on body \( i \). These active forces are acting at a collection of points generically denoted by \( \mathcal{U}_i \).

- Expression of this type of force acting at point \( U \in \mathcal{U}_i \):
  \[
  \mathbf{F}^a_U
  \]

- Virtual work produced by this set of forces:
  \[
  \sum_{U \in \mathcal{U}_i} [\delta \mathbf{r}_i^U]^T \cdot \mathbf{F}^a_U
  \]
Dealing with Active Torques

- Framework: We assume that a set of active torques acts on body $i$. We will assume that these active torques are acting at a collection of points generically denoted by $\mathcal{V}_i$.

- Expression of this type of torque acting at point $V \in \mathcal{V}_i$, expressed in the L-RF$_i$:

$$\mathbf{\bar{n}}_V^a$$

- Virtual work produced by this set of torques:

$$\sum_{V \in \mathcal{V}_i} [\delta \phi_i]^T \cdot \mathbf{\bar{n}}_V^a$$

- Comments: Note that since we are talking about rigid bodies, we have the same virtual rotation $\delta \phi_i$ no matter which of the torques acting on the rigid body we are dealing with
Virtual Work: Dealing with Constraint Reaction Forces

- Framework: We assume that a set of constraints acts on body $i$. These constraints most often lead to the presence of reaction forces. We will assume that the constraints on body $i$ are producing reaction (constraint) forces acting at a collection of points generically denoted by $Q_i$.

- Expression of this type of force acting at point $Q \in Q_i$:
  \[ F'_Q \]

- Virtual work produced by this set of forces:
  \[ \sum_{Q \in Q_i} [\delta r^Q_i]^T \cdot F'_Q \]

- Comments:
  - One of the outcomes of solving the EOM will be to compute the value of the reaction forces $F'_Q$ for $Q \in Q_i$
  - A separate discussion will follow on the meaning of the points $Q_i$
Virtual Work: Dealing with Constraint Reaction Torques

- Framework: We assume that a set of constraints acts on body $i$. These constraints might lead to the presence of reaction torques. We will assume that the constraints on body $i$ are producing reaction (constraint) torques acting at a collection of points generically denoted by $Z_i$.

- Expression of this type of torque acting at point $Z \in Z_i$ when represented in the L-RF (superscript $r$ stands for reaction):
  \[
  \mathbf{n}_Z^r
  \]

- Virtual work produced by these reaction torques:
  \[
  \sum_{Z \in Z_i} [\delta \phi_i]^T \cdot \mathbf{n}_Z^r
  \]

- Comments:
  - One of the outcomes of solving the EOM will be to compute the value of the reaction torques $\mathbf{n}_Z^r$ for $Z \in Z_i$
Deriving Newton’s Equations for a body with planar motion

- NOTE:
  - You should be able to derive Newton’s equations for a planar rigid body on your own (closed books)
  
  - Overall, the book does a very good job in explaining the derivation the equations of motion (EOM) for a rigid body

- No notes last time, the material is straight out of the book (page 200)

**Figure 6.1.1** Forces acting on a rigid body.
EOM: First Pass

- For now, assume that there are no concentrated forces

- Do this for *one* body for now

- We are going to deal with the distributed forces and use them in the context of d’Alembert’s Principle
  - Inertia forces
  - Internal forces
  - Other distributed forces (gravity)
On the meaning of $[\mathbf{B} \mathbf{s}^P]^T \cdot \mathbf{F}$

- In order to get the torque of $\mathbf{F}$ about the origin of the L-RF, I need to take the cross product of $\mathbf{s}^P \times \mathbf{F}$. I’m interested in the value of this torque.

- The value is $||\mathbf{s}^P|| \cdot ||\mathbf{F}|| \cdot \sin \theta$. What is an equivalent way to express this quantity?

- Use the identity
  
  $\sin \theta = \cos(\theta - \frac{\pi}{2})$

- Then, the value of the torque, is

  $n = ||\mathbf{s}^P|| \cdot ||\mathbf{F}|| \sin \theta$

  $= ||\mathbf{s}^P|| \cdot ||\mathbf{F}|| \cos(\theta - \frac{\pi}{2})$

  $= [\mathbf{s}^P \perp]^T \cdot \mathbf{F}$

  $= [\mathbf{B} \mathbf{s}^P]^T \cdot \mathbf{F}$

- To conclude, the value of the torque produced by $\mathbf{F}$ acting at point $P$ is

  $n = [\mathbf{B} \mathbf{s}^P]^T \cdot \mathbf{F}$
Example 6.1.1

- Tractor model: derive equations of motion

- Traction force $T_r$
- Small angle assumption (pitch angle $\phi$)
- Force in tires depends on tire deflection:

$$f(d) = \begin{cases} 
kd, & d \geq 0 \\
0, & d < 0 
\end{cases}$$

Figure 6.1.2 Plane motion of a tractor.
End: 6.1.3
Begin: 6.1.4
Inertia Properties of Rigid Bodies

- Issues discussed:
  - Determining location of the center of mass (centroid) of a rigid body
  - Parallel axis theorem
  - Mass moment of inertia of composite bodies
  - Location of centroid for a rigid body with a symmetry axis
Location of the Center of Mass

- By definition the centroid is that point of a body for which, when you place a reference frame at that point and compute a certain integral in relation to that reference frame, the integral vanishes:

\[
\int_{m} s^P \, dm(P) = 0
\]

Q: How can I determine the location of the center of mass?
Mass Moment of Inertia (MMI)

- The MMI was defined as:

\[ J' = \int_\mathcal{M} [s'P]^T s'P \ dm(P) \]

- Suppose that the reference frame with respect to which the integral is computed is a centroidal RF

- You might ask yourself, what happens if I decide to evaluate the integral above with respect to a different reference frame, located at a different point attached to this body?

- Answer is provided by the parallel axis theorem (\( \rho \) is vector from centroidal reference frame to the new reference frame):

\[ J'_{new} = J' + m (\rho^T \rho) \]
MMI of a Composite Body

- Step 1: Compute the centroid of the composite body

\[ \rho'' = \frac{1}{m} \sum_{i=1}^{k} m_i \rho''_i \]

- Step 2: For each sub-body, apply the parallel axis theorem to compute the MMI of that sub-body with respect to the newly computed centroid

\[ J^* = \sum_{i=1}^{k} \left( J'_i + m_i \rho_i^* T \rho_i^* \right) \]

- Note: if holes are present in the composite body, it’s ok to add and subtract material (this translates into positive and negative mass)

Figure 6.1.5  Body made up of subcomponents.
What can one say if the rigid body has a symmetry axis?

- Here symmetry axis means that both mass and geometry are symmetric with respect to that axis.

You can say this: the centroid is somewhere along that axis.

NOTE: if the rigid body has two axes of symmetry, the centroid is on each of them, and therefore is where they intersect.

Figure 6.1.4  Body with axis of symmetry.
Determining the Expression of Generalized Forces (6.2)