ME451
Kinematics and Dynamics of Machine Systems

Dynamics of Planar Systems

November 4, 2010

Chapter 6

Before we get started…

- Last Time
  - Midterm exam
  - Prior to exam: closed the book on the Kinematics Analysis

- Today
  - Start Chapter 6: Dynamics
    - Work on Sections 6.1.1 and 6.1.2

- Due Tuesday:
  - Take home exam
  - MATLAB Assignment
  - ADAMS Assignment

- Not much traffic on the forum…

Purpose of Chapter 6

- At the end of this chapter you should understand what “dynamics” means and how you should go about carrying out a dynamics analysis
- We’ll learn a couple of things:
  - How to formulate the equations that govern the time evolution of a system of bodies in planar motion
    - These equations are differential equations and they are called equations of motion
    - As many bodies as you wish, connected by any joints we’ve learned about…
  - How to compute the reaction forces in any joint connecting any two bodies in the mechanism
  - Understand how to properly handle the applied (external) forces to correctly use them in formulating the equations of motion

The Idea, in a Nutshell…

- First part of the class: **Kinematics**
  - You have as <u>many</u> constraints (kinematic and driving) as generalized coordinates
  - <u>No spare</u> degrees of freedom left
  - Position, velocity, acceleration found as the solution of algebraic problems (both nonlinear and linear)
  - We do not care whatsoever about forces applied to the system, we are told what the motions are and that's enough for the purpose of kinematics

- Second part of the class: **Dynamics**
  - You only have a <u>few</u> constraints imposed on the system
  - You have <u>extra</u> degrees of freedom
  - The system evolves in time as a result of external forces applied on it
  - We very much care about forces applied and inertia properties of the components of the mechanism (mass, mass moment of inertia)

Some clarifications

- Dynamics **<u>key</u>** question: how can I get the acceleration of each body of the mechanism?
  - Why is acceleration so relevant? If you know the acceleration you can integrate it twice to get velocity and position information for each body
  - How is the acceleration of a body "*i* "measured in the first place?
    - You attach a reference frame on body "*i* " and measure the acceleration of the body reference frame with respect to the global reference frame:

$$\ddot{\mathbf{q}}_i = \begin{bmatrix} \ddot{x}_i \\ \ddot{y}_i \\ \ddot{\phi}_i \end{bmatrix}$$

- The answer to the <u>key</u> question: To get the acceleration of each body, you first need to formulate the equations of motion
  - Remember **F=ma**?
    - Actually, the proper way to state this is **ma=F**, which is the "equation of motion", that is, what we are after here

Equations of motion of ONE planar RIGID body

- Framework:
  - We are dealing with <u>rigid</u> bodies
  - For this lecture, we'll consider only one body
    - We'll extend to more bodies next week…
- What are we after?
  - Proving that for one body with a **reference frame attached at its center of mass location** the equations of motion are:

$$\begin{aligned} m\ddot{\mathbf{r}} - \mathbf{F} &= \mathbf{0} \qquad \text{Equations of Motion governing translation} \\ J'\ddot{\phi} - n &= 0 \qquad \text{Equation of Motion governing rotation} \end{aligned}$$

r is the position of the body local reference frame

ϕ is the orientation of the body local reference frame

Equations of Motion (EOM)
Some clarifications…

- Centroidal reference frame of a body
  - A reference frame located right at the center of mass of that body
  - How is this special? It's special since a certain integral vanishes...

$$\int_m \mathbf{s}'^P \, d\mathrm{m}(P) = 0$$

- What is J' ?
  - Mass moment of inertia

$$J' = \int_m \left[\mathbf{s}'^P\right]^T \mathbf{s}'^P \, d\mathrm{m}(P)$$

NOTE: Textbook uses misleading notation

$$\mathbf{s}'^{PT} \Leftrightarrow \left[\mathbf{s}'^P\right]^T$$

Two Principles

- Principle of Virtual Work
  - Applies to a collection of particles
  - States that a configuration is an equilibrium configuration if and only if the virtual work of the forces acting on the collection of particle is zero

$$\sum_i \delta \mathbf{r}_i^T \cdot \mathbf{F}_i = 0$$

- D'Alembert's Principle
  - For a collection of particles <u>moving</u> around you can still fall back on the Principle of Virtual Work when you also include in the set of forces acting on each particle i its inertia force

$$\sum_i \delta \mathbf{r}_i^T \cdot (\mathbf{F}_i - m_i \ddot{\mathbf{r}}_i) = 0$$

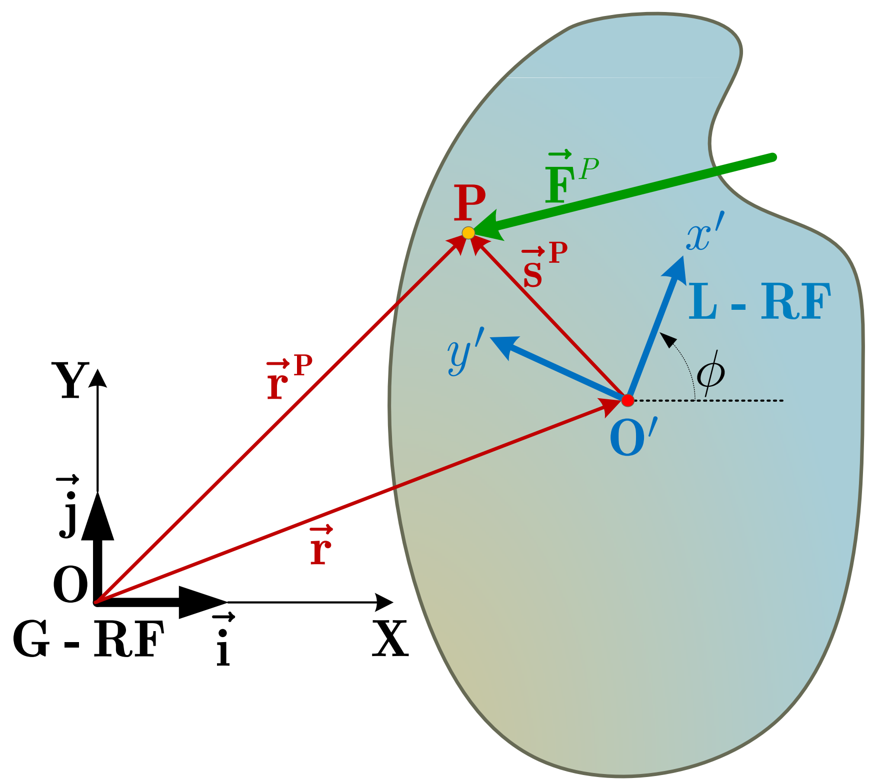
  - Note: we are talking here about a collection of *particles*
    - Consequently, we'll have to regard each rigid body as a collection of particles that are rigidly connected to each other and that together make up the body

Virtual Displacement and Virtual Work

- Imagine that force $\vec{\mathbf{F}}^P$ acts on a rigid body at point P. The virtual work done by this force is

$$\delta W^{F_P} = \left[\delta \mathbf{r}^P\right]^T \cdot \mathbf{F}^P$$

- Here $\delta \mathbf{r}^P$ represents a virtual displacement of the point P in response to a virtual translation and a virtual rotation experienced by the body to which P is attached.



[Small Detour, 3 slides]

The Variation $\delta\mathbf{u}$ of a Function $\mathbf{u}(\mathbf{q},t)$ due to a *small* change in its argument $\mathbf{q} \rightarrow \mathbf{q} + \delta\mathbf{q}$

- Framework: assume you have a vector quantity that depends on $\mathbf{q}$. Assume that the value of $\mathbf{q}$ changes to $\mathbf{q} + \delta\mathbf{q}$. What is the variation in the quantity that depends on $\mathbf{q}$ due to the said change?
- Specifically, assume the vector quantity of interest is $\mathbf{u}$, and $\mathbf{u}$ depends on $\mathbf{q}$ and possibly time t:

$$\mathbf{u} = \mathbf{u}(\mathbf{q}, t)$$

- I am interested at a fixed time t in the $\delta\mathbf{u}$ below given $\mathbf{q}$, $\delta\mathbf{q}$, and the expression of $\mathbf{u}(\mathbf{q})$:

$$\mathbf{q} \longrightarrow \mathbf{u}(\mathbf{q}, t) \qquad \mathbf{q} + \delta\mathbf{q} \longrightarrow \mathbf{u}(\mathbf{q} + \delta\mathbf{q}, t) = \mathbf{u}(\mathbf{q}, t) + \delta\mathbf{u}$$

$$\delta\mathbf{u} = ?$$

$\delta \mathbf{u} = ?$

- Without getting into details (more on this in ME751 or in a Calculus of Variations class/textbook)

$$\delta \mathbf{u} = \mathbf{u}_{\mathbf{q}} \cdot \delta \mathbf{q}$$

- Likewise, given two functions $\mathbf{f}(\mathbf{q}, t)$ and $\mathbf{g}(\mathbf{q}, t)$, we have that

$$\delta(\mathbf{f} + \mathbf{g}) = \delta \mathbf{f} + \delta \mathbf{q} \qquad \delta(\mathbf{f} - \mathbf{g}) = \delta \mathbf{f} - \delta \mathbf{q}$$

[Example]

Calculus of Variations

- Indicate the change in the quantities below that are a consequence of applying a virtual displacement δ**q** to the generalized coordinates **q**

Assumptions:
$\mathbf{g} = \mathbf{g}(\mathbf{q})$
$\mathbf{p} = \mathbf{p}(\mathbf{q})$
$\mathbf{C}$ - constant matrix

$$\mathbf{h}(\mathbf{q}) = \mathbf{C}\mathbf{q}$$

$$h(\mathbf{q}) = \mathbf{g}^T\mathbf{p}$$

$$h(\mathbf{q}) = \mathbf{p}^T\mathbf{C}\mathbf{q}$$

The dimensions of the vectors and matrix above such that all the operations listed can be carried out.

Calculus of Variations in ME451

- In our case we are interested in variations of kinematic quantities (locations of a point P, of **A** matrix, etc.) due to a variations in the location and orientation of a body.

- Variation in location of the L-RF:

$$\mathbf{r} \longrightarrow \mathbf{r} + \delta\mathbf{r}$$

- Variation in orientation of the L-RF:

$$\phi \longrightarrow \phi + \delta\phi$$

- As far as the change of orientation matrix $\mathbf{A}(\phi)$ is concerned, using the result stated two slides ago, we have that a variation in the orientation leads to the following variation in **A**:

$$\delta\mathbf{A} = \frac{d\mathbf{A}}{d\phi}\,\delta\phi = \mathbf{B}\,\delta\phi$$

Calculus of Variations in ME451

Virtual Displacement of a Point P Attached to a Body

- Original position of P:

$$\mathbf{r}^P = \mathbf{r} + \mathbf{A}\bar{\mathbf{s}}^P$$

- Position of P after the small change in the position and orientation of the rigid body:

$$\mathbf{r}^P + \delta\mathbf{r}^P = (\mathbf{r} + \delta\mathbf{r}) + (\mathbf{A} + \delta\mathbf{A})\bar{\mathbf{s}}^P$$

- Net change in position of point P:

$$\delta\mathbf{r}^P = \underbrace{(\mathbf{r}^P + \delta\mathbf{r}^P)}_{\text{Location, after Virtual Displacement}} - \underbrace{\mathbf{r}^P}_{\text{Location, Original}} = \delta\mathbf{r} + \delta\mathbf{A}\;\bar{\mathbf{s}}^P = \delta\mathbf{r} + \mathbf{B}\;\bar{\mathbf{s}}^P\delta\phi$$

- Matrix-Vector Form:

$$\delta\mathbf{r}^P = \left[\mathbf{I}_{2\times 2} \quad , \quad \mathbf{B}\;\bar{\mathbf{s}}^P\right] \cdot \begin{bmatrix} \delta\mathbf{r} \\ \delta\phi \end{bmatrix}$$

Deriving the EOM

Some Clarifications

- Assumptions:
  - All bodies that we work with are rigid*
  - The bodies undergo planar motion
- Start from scratch, that is, from the dynamics of a material point, work our way up to a body, then to a collection of bodies that are interacting through kinematic joints and/or friction & contact
- Derivation that follows next is also in the textbook

Some Clarifications

[regarding the "Rigid Body" concept]

- **Remark 1**: For a rigid body, the distance between two internal points P and R of the body does not change in time:

$$(\mathbf{r}^P - \mathbf{r}^R)^T \cdot (\mathbf{r}^P - \mathbf{r}^R) = \text{const.}$$

  - Then,

$$(\mathbf{r}^P - \mathbf{r}^R)^T \cdot \delta(\mathbf{r}^P - \mathbf{r}^R) = 0$$

- **Remark 2**: For a rigid body, any internal force $\mathbf{f}(P, R)$ acting between points P and R is along the direction defined by these two points. That is, with k a scalar that might depend on time,

$$\mathbf{f}(P, R) = k(\mathbf{r}^P - \mathbf{r}^R)$$

- Based on the previous two equations, one can conclude that

$$\mathbf{f}^T(P, R) \cdot \delta(\mathbf{r}^P - \mathbf{r}^R) = \left[\delta(\mathbf{r}^P - \mathbf{r}^R)\right]^T \cdot \mathbf{f}(P, R) = 0$$

- In plain English, the virtual work of the internal forces is zero. We'll revisit/need this shortly.

Road Map [2 weeks]

- Introduce the forces present in a mechanical system
  - Distributed
  - Concentrated

- Express the virtual work produced by each of these forces

- Apply principle of virtual work and obtain the EOM

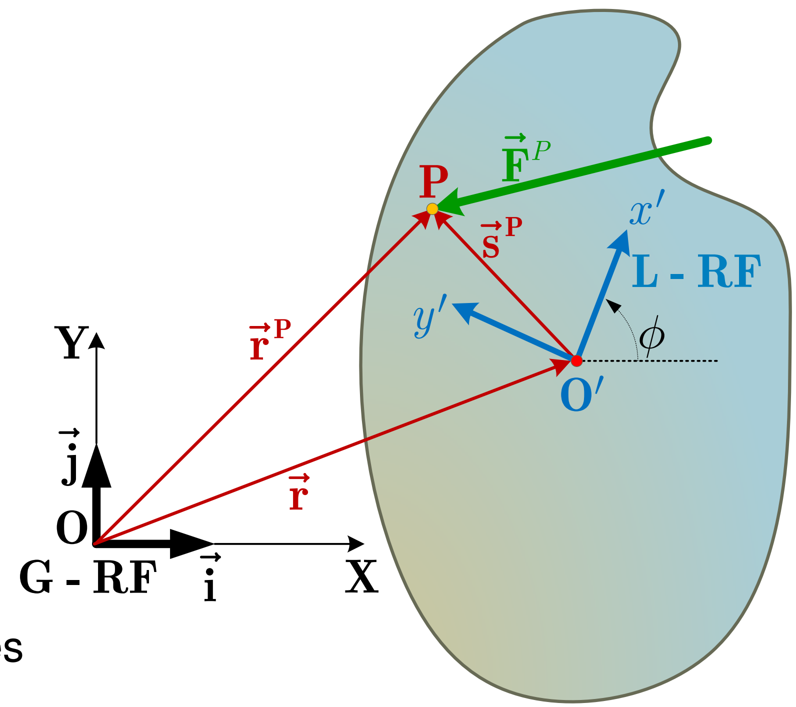
- Eliminate the reaction forces from the expression of the virtual work

- Obtain the constrained EOM (Newton-Euler form)

- Express the reaction (constraint) forces from the Lagrange multipliers

Types of Forces & Torques Acting on a Body

- Type 1: Distributed over the volume of a body
  - Inertia forces
  - Distributed forces
  - Internal interaction forces
- Type 2: Concentrated at a point
  - Action (or applied, or external) forces and torques
  - Reaction (or constraint) forces and torques



Virtual Work: Dealing with Inertia Forces

- Framework: we are considering a point P of body i. This point is associated with an infinitesimal mass element $dm_i(P)$
- Expression of the force:

$$-\ddot{\mathbf{r}}_i^P \; dm_i(P)$$

- Virtual work produced:

$$[\delta \mathbf{r}_i^P]^T \cdot [-\ddot{\mathbf{r}}_i^P \; dm_i(P)]$$

- Comments:
  - The total virtual work produced by this type of force is obtained by summing over all points of body i:

$$\int_{m_i} -[\delta \mathbf{r}_i^P]^T \cdot \ddot{\mathbf{r}}_i^P \; dm_i(P)$$

Virtual Work: Dealing with Mass-Distributed Forces

- Framework: we are considering a point P of body i. This point is associated with an infinitesimal mass element $dm_i(P)$. A force per unit mass, $\mathbf{f}_i(P)$, is assumed to act at point P.

- Expression of the force:

$$\mathbf{f}_i(P)\, dm_i(P)$$

- Virtual work produced:

$$[\delta \mathbf{r}_i^P]^T \cdot \mathbf{f}_i(P)\, dm_i(P)$$

- Comments:
  - The total virtual work produced by this type of force is obtained by summing over all points of body i:

$$\int_{m_i} [\delta \mathbf{r}_i^P]^T \cdot \mathbf{f}_i(P)\, dm_i(P)$$

  - This type of force is rarely seen in classical multibody dynamics. Exception: the force due to the gravitational field, which leads to the weight of the body. In this case $\mathbf{f}_i(P) = \mathbf{g}$, where $\mathbf{g}$ is the gravitational acceleration of magnitude $g \approx 9.81 \frac{m}{s^2}$ (in Madison, WI).