Before we get started...

- Last Time
  - Discussed driving constraints
    - A set of $ndof$ independent drivers specified to “occupy” all the degrees of freedom
    - Driver constraints
      - Absolute: $x$, $y$, $\phi$
      - Relative: distance, motion on a revolute joint, motion on a translational joint

- Today:
  - Excavator example, setting up driving constraints
  - Kinematic Analysis Wrap-up: Position, Velocity, and Acceleration Stages
  - Start wrecker-boom example

- Final Project proposal due in one week

- Midterm exam coming up on November 2
  - Review for exam: Monday, Nov 1, starting at 6 PM (precise room/time TBA)
Example: Specifying Relative Distance Drivers

- Generalized coordinates: $\mathbf{q} = [\phi_1, x_2, y_2, \phi_2]^T$

- Motions prescribed:
  
  \[ C_{41}(t) = \frac{1}{5} t + 1.8 \]
  \[ C_{12}(t) = \frac{1}{10} t + 1.9 \]

- Derive the constraints acting on system
- Derive linear system whose solution provides velocities $\mathbf{\dot{q}} = [\dot{\phi}_1, \dot{x}_2, \dot{y}_2, \dot{\phi}_2]^T$

Figure 3.5.6  Excavator boom assembly with two distance drivers.
Mechanism Analysis: Steps

- **Step A**: Identify *all* physical joints and drivers present in the system

- **Step B**: Identify the corresponding constraint equations $\Phi(q, t)$

- **Step C**: Solve for the Position as a function of time ($\Phi_q$ is needed)

- **Step D**: Solve for the Velocities as a function of time ($\nu$ is needed)

- **Step E**: Solve for the Accelerations as a function of time ($\gamma$ is needed)
Position, Velocity, and Acceleration Analysis (Section 3.6)

- The position analysis [Step C]:
  - It’s the tougher of the three
  - Requires the solution of a system of nonlinear equations
  - What you are after is determining at each time the location and orientation of each component (body) of the mechanism

- The velocity analysis [Step D]:
  - Requires the solution of a linear system of equations
  - Relatively simple
  - Carried out after you are finished with the position analysis

- The acceleration analysis [Step E]:
  - Requires the solution of a linear system of equations
  - What is challenging is generating the RHS of acceleration equation, $\gamma$
  - Carried out after you are finished with the velocity analysis
Position Analysis

- **Framework:**
  - Somebody presents you with a mechanism and you select the set of \( nc \) generalized coordinates to position and orient each body of the mechanism:
    \[
    \mathbf{q} = [x_1, y_1, \phi_1, x_2, y_2, \phi_2, \ldots]^T \in \mathbb{R}^{nc}
    \]
  - You inspect the mechanism and identify a set of \( nk \) kinematic constraints that must be satisfied by your coordinates:
    \[
    \Phi^K(\mathbf{q}) = 0
    \]
  - Next, you identify the set of \( nd \) driver constraints that move the mechanism:
    \[
    \Phi^D(\mathbf{q}, t) = 0
    \]

**NOTE:** YOU MUST HAVE \( nc = nk + nd \)
We end up with this problem: given a time \( t \), find that set of generalized coordinates \( q \) that satisfy the equations:

\[
\Phi(q, t) = \begin{bmatrix} \Phi^K(q) \\ \Phi^D(q, t) \end{bmatrix} = 0
\]

What’s the idea here?
- Set time \( t=0 \), and find a solution \( q \) by solving above equations
- Next, set time \( t=0.001 \) and find a solution \( q \) by solving above equations
- Next, set time \( t=0.002 \) and find a solution \( q \) by solving above equations
- Next, set time \( t=0.003 \) and find a solution \( q \) by solving above equations
- …
- Stop when you reach the end of the interval in which you are interested in the position

What you do is find the time evolution on a time grid with step size \( \Delta t=0.001 \)
- You can then plot the solution as a function of time and get the time evolution of your mechanism
Two issues with the described methodology for finding the time evolution of the mechanism:

- The equations that we have to solve at each time $t$ are nonlinear, so a first hurdle is being able to solve this nonlinear system.
  - Deal with this issue later (next week, Newton-Raphson method).

- The second issue comes when you start thinking about the solution that you’ve just got using a numerical algorithm.
  - How do you know that what you got is a meaningful thing?
    - Remember, a nonlinear system can have an arbitrary number of solutions.
  - Deal with this issue now.
Position Analysis: Implicit Function Theorem

- Is the solution of our nonlinear system well behaved? A sufficient condition is provided by the Implicit Function Theorem.

- In layman’s words, this is what the theorem says:
  - Let’s say that we are at some time $t_k$, and we just found the solution $q_k$ and we question the quality of this solution.
  - If the constraint Jacobian is nonsingular in this configuration, that is,
    $$\det |\Phi_q(q_k, t_k)| \neq 0$$
  - … then, we can conclude that the solution is unique, and not only at $t_k$, but in a small interval $\delta$ about time $t_k$.
  - Additionally, in this small time interval, there is an explicit functional dependency of $q$ on $t$, that is, there is a function $f(t)$ such that:
    $$q(t) = f(t) \quad \text{for} \quad |t - t_k| < \delta$$
End Position Analysis

Begin Velocity Analysis
Velocity Analysis

- This is simple. What is the framework?

- You just found \( q \) at time \( t \), that is, the location and orientation of each component of the mechanism at time \( t \), and now you want to find the velocity of each component (body) of the mechanism.

- Taking one time derivative of the constraints leads to the velocity equation:

\[
\Phi(q, t) = 0 \quad \Rightarrow \quad \dot{\Phi}(q, t) = 0 \quad \Leftrightarrow \quad \Phi_q(q, t) \cdot \dot{q} = \nu
\]

- In layman’s words, once you have \( q^{(k)} \) you can find \( \dot{q}^{(k)} \) at time \( t_k \) by solving the linear system

\[
\Phi_q(q^{(k)}, t_k) \cdot \dot{q}^{(k)} = \nu^{(k)}
\]
Velocity Analysis

- **Notation:** please note the subscript is in parentheses
  \[ q(k), \dot{q}(k) \]
  - It indicates that that quantity is evaluated at \( t_k \)
  - If no parentheses, can be mistaken for the coordinates associated with body “k”

- **Observations:**
  - Note that as long as the constraint Jacobian is nonsingular, you can solve this linear system and recover the velocity \( \dot{q}(k) \)
  - The reason velocity analysis is easy is that, unlike for position analysis where you have to solve a nonlinear system, now you are dealing with a linear system, which is easy to solve
  - Note that the velocity analysis comes after the position analysis is completed, and you are in a new configuration of the mechanism in which you are about to find out its velocity
End Velocity Analysis
Begin Acceleration Analysis
Acceleration Analysis

- This is also fairly simple. What is the framework?

- You just found \( q(k) \) and \( \dot{q}(k) \) at time \( t_k \), that is, where the mechanism is at time \( t_k \), and what its velocity is.

- You’d like to know the acceleration of each component of the model.

- Taking two time derivatives of the constraints leads to the acceleration equation:

\[
\Phi(q(k), t_k) = 0 \quad \Rightarrow \quad \ddot{\Phi}(q(k), t_k) = 0 \quad \Leftrightarrow \quad \Phi_q(q(k), t_k) \cdot \ddot{q}(k) = \gamma(k)
\]
In other words, you find the acceleration (second time derivative of $q$ at time $t_k$) as the solution of a linear system:

$$\Phi_q(q(k), t_k) \cdot \ddot{q}(k) = \gamma(k)$$

Observations:

- The equation above illustrates why we have been interested in the expression of $\gamma$, the RHS of the acceleration equation:

$$\gamma = -(\Phi_q \dot{q}) q \ddot{q} - 2\Phi_q t \dot{q} - \Phi_{tt}$$

- Note that you again want the constraint Jacobian to be nonsingular, since then you can solve the acceleration linear system and obtained the acceleration $\ddot{q}(k)$
We looked at the KINEMATICS of a mechanism

That is, we are interested in how this mechanism moves in response to a set of kinematic drives (motions) applied to it

What one has to do:

- **Step A**: Identify *all* physical joints and drivers present in the system
- **Step B**: Identify the corresponding constraint equations \( \Phi(q, t) \)
- **Step C**: Solve for the Position as a function of time (\( \Phi_q \) is needed)
- **Step D**: Solve for the Velocities as a function of time (\( \nu \) is needed)
- **Step E**: Solve for the Accelerations as a function of time (\( \gamma \) is needed)