Before we get started...

- Last time:
  - Revolute Joint
  - Translational Joint

- Today: Move on and cover a couple of other constraints
  - Composite joints:
    - distance constraint
    - revolute-translational
  - Gears – I’ll skip these
  - Cam-follower
  - Point follower

- Next week:
  - ADAMS Tutorial, part two. No lecture on Tu and Th
  - Focus is on ADAMS co-simulation and interfacing to MATLAB for mechatronics
  - No office hours, I’ll be traveling the entire week
Composite Joints (CJ)

- Just a means to eliminate one intermediate body whose kinematics you are not interested in

- Revolute-Revolute CJ
  - Also called a coupler
  - Practically eliminates need of connecting rod
  - Given to you (joint attributes):
    - Location of points $P_i$ and $P_j$
    - Distance $d_{ij}$ of the massless rod

- Revolute-Translational CJ
  - Given to you (joint attributes):
    - Distance $c$
    - Point $P_j$ (location of revolute joint)
    - Axis of translation $v_i'$
    - $v_i'$ is assumed to be a unit vector
Composite Joints

- One follows exactly the same steps as for any joint:
  - Step 1: Physically, what type of motion does the joint allow?
  - Step 2: Constraint Equations $\Phi(q,t) = ?$
  - Step 3: Constraint Jacobian $\Phi_q = ?$
  - Step 4: $v = ?$
  - Step 5: $\gamma = ?$
Moving on to gears (section 3.4)
Gears

- Convex-convex gears
  - Gear teeth on the periphery of the gears cause the pitch circles shown to roll relative to each other, without slip
- First Goal: find the angle $\theta$, that is, the angle of the carrier

- What’s known:
  - Angles $\theta_i$ and $\theta_j$
  - The radii $R_i$ and $R_j$

- You need to express $\theta$ as a function of these four quantities plus the orientation angles $\phi_i$ and $\phi_j$

- Kinematically: $P_iP_j$ should always be perpendicular to the contact plane

![Figure 3.4.2](image) Geometry of gear set.
Gears - Discussion of Figure 3.4.2 (Geometry of gear set)

$O^i_{i}x'_iy'_i$ represents the local reference frame attached to gear $i$. This reference frame is rotated by an angle $\phi_i$ with respect to the global reference frame. This angle $\phi_i$ depends on time, and changes as the attitude (orientation) of body $i$ changes.

$O^j_{j}x'_jy'_j$ represents the local reference frame attached to gear $j$. This reference frame is rotated by an angle $\phi_j$ with respect to the global reference frame. This angle $\phi_j$ depends on time, and changes as the attitude (orientation) of body $j$ changes.

When the gears were assembled, the points $Q_i$ and $Q_j$ were the contact points between the two gears (that’s where the gears came in contact for the first time, at time $t = 0$). If one gear is activated and it starts rotating, the second gear will follow and the two points $Q_i$ and $Q_j$ will separate and follow their destiny, away from the contact point which is denoted by $D$.

There is an angle $\alpha_i$ that indicates how far the original contact point, $Q_i$, is from the current contact point, $D$. By the same token, there is an angle $\alpha_j$ that indicates how far the original contact point, $Q_j$, is from the current contact point, $D$.

Since the local reference frame $O^i_{i}x'_iy'_i$ is attached to gear $i$, and the point $Q_i$ is also attached to gear $i$, the angle that positions $Q_i$ in $O^i_{i}x'_iy'_i$ stays constant at all times, and it’s denoted by $\theta_i$. For gear $j$ a similar angle $\theta_j$ is defined.
Gears - Discussion of Figure 3.4.2
(Geometry of gear set)

The contact point $D$ is always on the line that connects $P_i$ and $P_j$, the centers of the two gears, respectively.

What we are after is determining the angle $\theta$ that implicitly defines the perpendicular on the plane of contact (the plane that goes through $D$ and is tangent to the two gears at point $D$). Once this angle $\theta$ becomes available, the vector that is perpendicular on the tangent plane assumes the expression:

$$u^\perp = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}.$$

Using Figure 3.4.2, after performing some manipulations, the angle $\theta$ is expressed as

$$\theta = \frac{R_i(\phi_i + \theta_i) + R_j(\phi_j + \theta_j - \pi)}{R_i + R_j}. \tag{1}$$

The kinematic constraint associated with the gear set requires that the vectors $P_i \hat{P}_j$ and $\bar{u}$ are parallel, or in other words,

$$\Phi^{(i,j)} = (r^P_j - r^P_i)^T \cdot u^\perp = 0 \tag{2}$$

Note that the important thing is that this angle $\theta$ depends on the value of $\phi_i$ and $\phi_j$, which in turn depend on the orientation of the two gears. What Eq. (2) is telling us is that $\phi_i$ and $\phi_j$ can not be arbitrarily changing. Rather, as they change in time, they should change in such a way so that the angle $\theta$ computed with Eq. (1) will satisfy the condition of Eq. (2).

Note: there are a couple of mistakes in the book, see Errata slide before 8
Example: 3.4.1

- Gear 1 is fixed to ground
- Given to you: $\phi_1 = 0$, $\theta_1 = \pi/6$, $\theta_2 = 7\pi/6$, $R_1 = 1$, $R_2 = 2$
- Find $\phi_2$ as gear 2 falls to the position shown (carrier line $P_1P_2$ becomes vertical)
Gears (Convex-Convex)

- Convex-concave gears – we are not going to look into this class of gears
- The approach is the same, that is, expressing the angle $\theta$ that allows on to find the angle of the
- Next, a perpendicularly condition using $\mathbf{u}$ and $P_iP_j$ is imposed (just like for convex-convex gears)

Figure 3.4.4 Concave–convex gear set.
Example: 3.4.1

- Gear 1 is fixed to ground
- Given to you: $\phi_1 = 0$, $\theta_1 = \pi/6$, $\theta_2 = 7\pi/6$, $R_1 = 1$, $R_2 = 2$
- Find $\phi_2$ as gear 2 falls to the position shown (carrier line $P_1P_2$ becomes vertical)
Rack and Pinion Preamble

- **Framework:**
  - Two points $P_i$ and $Q_i$ on body $i$ define the rack center line.
  - Radius of pitch circle for pinion is $R_j$.
  - There is no relative sliding between pitch circle and rack center line.
  - $Q_i$ and $Q_j$ are the points where the rack and pinion were in contact at time $t=0$.

- **NOTE:**
  - A rack-and-pinion type kinematic constraint is a limit case of a pair of convex-convex gears.
    - Take the radius $R_i$ to infinity, and the pitch line for gear $i$ will become the rack center line.
Rack and Pinion Kinematics

- Kinematic constraints that define the relative motion:
  - At any time, the distance between the point $P_j$ and the contact point $D$ should stay constant (this is equal to the radius of the gear $R_j$)
  - The length of the segment $Q_iD$ and the length of the arc $Q_jD$ should be equal (no slip condition)

- Rack-and-pinion removes two DOFs of the relative motion between these two bodies

**Figure 3.4.5** Rack and pinion.
Rack and Pinion Pair

- Step 1: Understand the physical element
- Step 2: Constraint Equations $\Phi(q,t) = ?$
- Step 3: Constraint Jacobian $\Phi_q = ?$
- Step 4: $\nu = ?$
- Step 5: $\gamma = ?$
End Gear Kinematics

Begin Cam-Follower Kinematics
Preamble: Boundary of a Convex Body

- Assumption: the bodies we are dealing with are convex
  - To any point on the boundary corresponds one value of the angle $\alpha$ (this is like the yaw angle, see figure below)

- The distance from the reference point $Q$ to any point $P$ on the convex boundary is a function of $\alpha$:
  $$||PQ|| = \rho(\alpha)$$

- It all boils down to expressing two quantities as functions of $\alpha$
  - The position of $P$, denoted by $r_P$
  - The tangent at point $P$, denoted by $g$
Cam-Follower Pair

- Assumption: no chattering takes place
- The basic idea: two bodies are in contact, and at the contact point the two bodies share:
  - The contact point
  - The tangent to the boundaries

Recall that a point is located by the angle $\alpha_i$ on body $i$, and $\alpha_j$ on body $j$.

Therefore, when dealing with a cam-follower, in addition to the $x, y, \theta$ coordinates for each body one needs to rely on one additional generalized coordinate, namely the contact point angle $\alpha$:
- Body $i$: $x_i, y_i, \phi_i, \alpha_i$
- Body $j$: $x_j, y_j, \phi_j, \alpha_j$
Cam-Follower Constraint

● Step 1: Understand the physical element

● Step 2: Constraint Equations $\Phi(q,t) = ?$

● Step 3: Constraint Jacobian $\Phi_q = ?$

● Step 4: $v = ?$

● Step 5: $\gamma = ?$
Example

- Determine the expression of the tangents $g_1$ and $g_2$

$$\rho_1(\alpha_1) = \begin{cases} -\frac{1}{4} \cos 3\alpha_1 + \frac{5}{4} & \text{if } 0 \leq \alpha_1 < \frac{2\pi}{3} \\ 1 & \text{if } \frac{2\pi}{3} \leq \alpha_1 \leq 2\pi \end{cases}$$

$$\rho_2(\alpha_2) = \frac{1}{4}$$
Cam flat-faced-follower Pair

- A particular case of the general cam-follower pair
  - Cam stays just like before
  - Flat follower
  - Typical application: internal combustion engine
  - Not covered in detail, HW touches on this case

*Figure 3.4.10* Cam–flat-faced follower pair.

*Figure 3.4.11* Cam–flat-faced follower in an internal combustion engine.
Point-Follower Pair

- Framework (Step 1):
  - Pin P is attached to body i and can move in slot attached to body j.
  - NOTE: the book forgot to mention what $g_j$ is (pp.85, eq. 3.4.32)
    - It represents the tangent to the slot in which P is allowed to move

- The location of point P in slot attached to body j is captured by angle $\alpha_j$ that parameterizes the slot.

- Therefore, when dealing with a point-follower we’ll be dealing with the following set of generalized coordinates:
  - Body i: $x_i$, $y_i$, $\phi_i$,
  - Body j: $x_j$, $y_j$, $\phi_j$, $\alpha_j$

Figure 3.4.12  Point–follower pair.
Point-Follower Pair

- Step 2: Constraint Equations $\Phi(q, t) = ?$
- Step 3: Constraint Jacobian $\Phi_q = ?$
- Step 4: $\nu = ?$
- Step 5: $\gamma = ?$