ME451
Kinematics and Dynamics of Machine Systems

Vel. And Acc. of a Fixed Point in Moving Frame - 2.6

Basic Concepts in Planar Kinematics - 3.1

September 16, 2010

Before we get started…

- Last Time
  - We discussed about taking derivatives
    - Partial Derivatives (sensitivities)
      - Recall the “accordion rule”
    - Chain rule of taking partial derivatives
    - Time Derivatives
    - These are going to come up time and again in this class

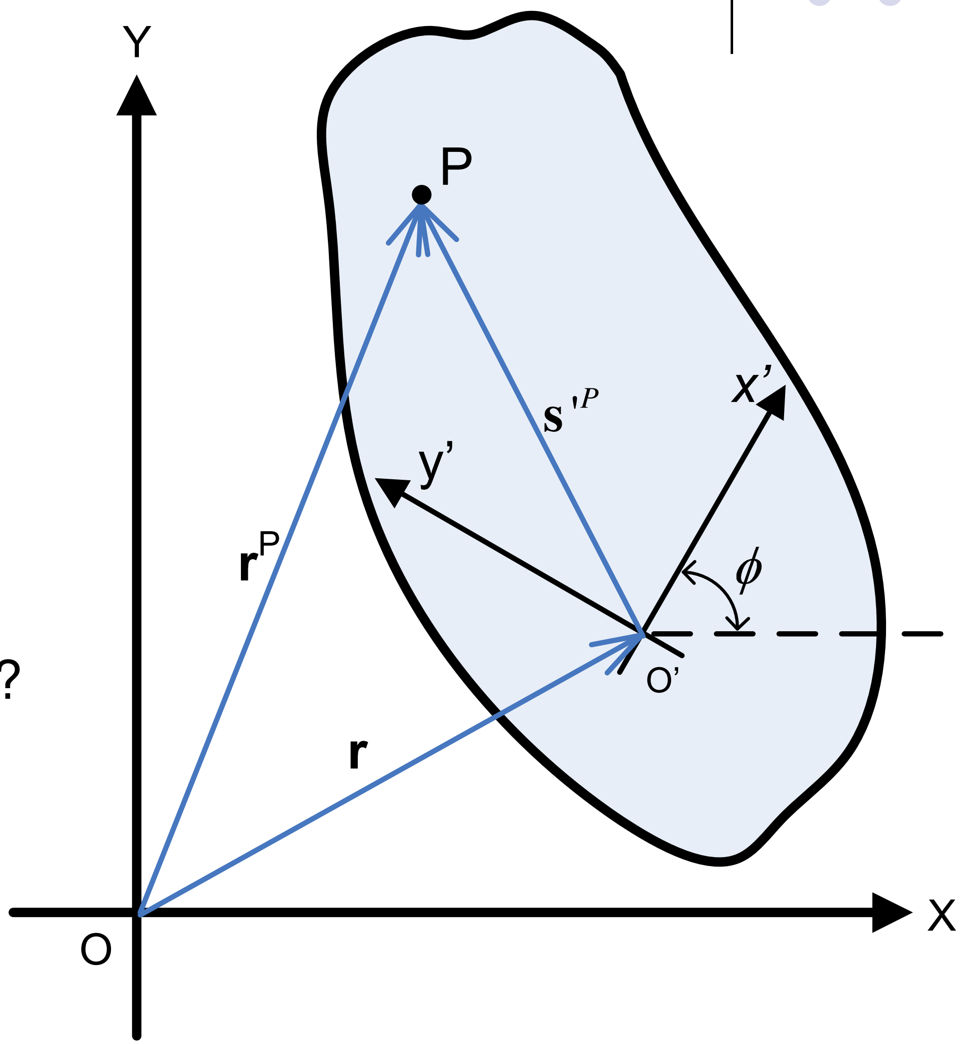
- Today
  - Wrap up Chapter 2
  - Start covering material in Chapter 3: Kinematics Analysis

- Next week, **<u>both</u>** lectures:
  - ADAMS tutorial: TA will email to you location of the lab

Section 2.6

Velocity of a Point Fixed to a Moving Frame (2.6)

- The problem at hand:
  - Rigid body, colored in blue
  - This body moves in space, and the location of a point P attached to this body is a function of time and is represented by $\mathbf{r}^P(t)$
- Question: What is the velocity of P ?
  - Equivalently, what is the time derivative of $\mathbf{r}^P(t)$



Velocity of a Point Fixed to a Moving Frame

- Something to keep in mind: we'll manipulate quantities that depend on the generalized coordinates, which in turn depend on time
  - Specifically, the orientation matrix **A** depends on the generalized coordinate ϕ , which depends on t

- This is where the [time and partial] derivates discussed before come into play

$$\mathbf{r}^P = \mathbf{r} + \mathbf{A}\mathbf{s}'^P$$

$$\Downarrow \quad \text{(Take time derivative)}$$

$$\dot{\mathbf{r}}^P = \dot{\mathbf{r}} + \dot{\mathbf{A}}\mathbf{s}'^P = \dot{\mathbf{r}} + \dot{\phi}\mathbf{B}\mathbf{s}'^P$$

Matrices of Interest

- Rotation Matrix **R**: $$\mathbf{R} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$
  - Note that when applied to a vector, this rotation matrix produces a new vector that is perpendicular to the original vector (counterclockwise rotation)

- The **B** matrix is always defined in conjunction with an **A** matrix. By definition
$$\mathbf{B} \equiv \frac{d\mathbf{A}}{d\phi}$$
  - The **B** matrix
$$\mathbf{B} = \begin{bmatrix} -\sin\phi & -\cos\phi \\ \cos\phi & -\sin\phi \end{bmatrix}$$
  - Note the following three formulas:

$$\mathbf{B} = \mathbf{AR} = \mathbf{RA} \qquad \frac{d\mathbf{B}}{d\phi} = -\mathbf{A} \qquad \forall \mathbf{v}, \quad \mathbf{v}^{\perp} = \mathbf{R}\mathbf{v}$$

Acceleration of a Fixed Point in a Moving Frame

- Same idea as for velocity, except that you need two time derivates to get accelerations

$$\dot{\mathbf{r}}^P = \dot{\mathbf{r}} + \dot{\mathbf{A}}\mathbf{s}'^P = \dot{\mathbf{r}} + \dot{\phi}\mathbf{B}\mathbf{s}'^P$$

$$\Downarrow \qquad \text{(Take time derivative)}$$

$$\ddot{\mathbf{r}}^P = \ddot{\mathbf{r}} + \ddot{\phi}\mathbf{B}\mathbf{s}'^P - \dot{\phi}^2\mathbf{A}\mathbf{s}'^P$$

Exercise:

- You are given:
  - Position of point P in LRF

  - Position of LRF
  - Orientation of LRF
  - Translational velocity of LRF
  - Angular velocity of LRF
  - Translational acceleration of LRF
  - Angular acceleration of LRF

$$\mathbf{s}'^P = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\mathbf{r} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \phi = \frac{\pi}{3}$$

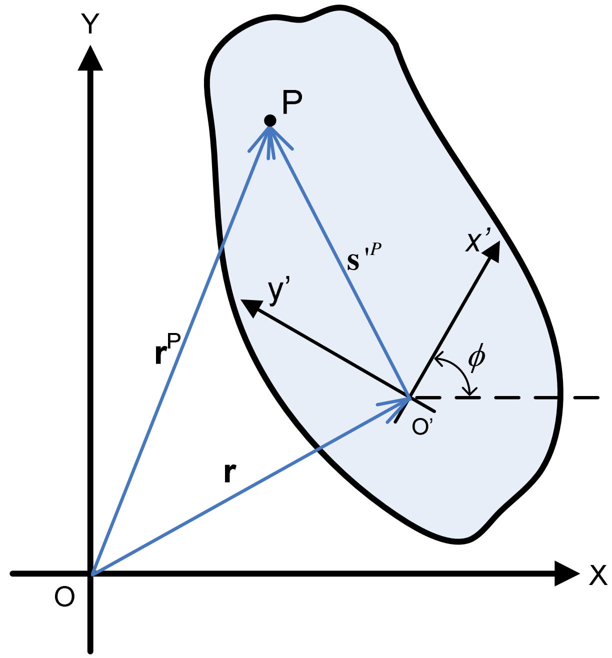
$$\dot{\mathbf{r}} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \qquad \dot{\phi} \equiv \omega = 3$$

$$\ddot{\mathbf{r}} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \qquad \ddot{\phi} \equiv \dot{\omega} = 5$$

Nomenclature:

- LRF – local reference frame
- GRF – global reference frame

- You are asked:
  - The position of P in GRF
  - The velocity of P in GRF
  - The acceleration of P in GRF



$$\mathbf{r}^P = ?$$

$$\dot{\mathbf{r}}^P = ?$$

$$\ddot{\mathbf{r}}^P = ?$$

End Chapter 2
Begin Chapter 3
(Kinematics)

What is Kinematics?

- Study of the position, velocity, and acceleration of a system of interconnected bodies that make up a mechanism, <u>independent</u> of the forces that produce the motion

Why is Kinematics Important?

- It can be an end in itself…
  - *Kinematic Analysis* - Interested how components of a certain mechanism move when motion[s] are applied
  - *Kinematic Synthesis* – Interested in finding how to design a mechanism to perform a certain operation in a certain way
    - NOTE: we only focus on Kinematic Analysis

- It is also an essential ingredient when formulating the Kinetic problem (so called Dynamics Analysis, discussed in Chapter 6)

- In general, people are more interested in the Dynamic Analysis rather than in the Kinematic Analysis

Purpose of Rest of Lecture

- Before getting lost in the details of Kinematics Analysis:
  - Present a collection of terms that will help understand the “language” of Kinematics
  - Give a 30,000 feet perspective of things to come and justifies the need for the material presented over the next 2-3 weeks

- Among the concepts introduced today, here are the more important ones:
  - Constraint equations (as a means to defining the geometry associated with the motion of a mechanism)
  - Jacobian matrix (or simply, the Jacobian)

Nomenclature

- Rigid body
- Body-fixed Reference Frame (also called Local Reference Frame, LRF)
- Generalized coordinates $$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ \cdots \\ q_{nc} \end{bmatrix} \in \mathbb{R}^{nc}$$
- Cartesian generalized coordinates $$\mathbf{q}_i = \begin{bmatrix} x \\ y \\ \phi \end{bmatrix}_i$$
  - NOTE: for a mechanism with nb bodies, the number of Cartesian generalized coordinates associated is

$$nc = 3 \cdot nb$$

Constraints

- What are they, and what role do they play?
  - A collection of equations that if satisfied, they coerce the bodies in the model to move like the bodies of the mechanism
- Most important thing in relation to constraints:
  - For each joint in the model, the equations of constraint that you use must imply the relative motion allowed by the joint
  - Keep in mind: the way you **model** should resemble the **physical system**
- Taxonomy of constraints:
  - Holonomic vs. Nonholonomic constraints
  - Scleronomic vs. Rheonomic constraints

Degrees of Freedom

- Number of degrees of freedom (NDOF, ndof) is equal to total number of generalized coordinates minus constraints that these coordinates must satisfy

- Is an attribute of the model, and it is independent of generalized coordinates used to represent the time evolution of the mechanism

- In general, for carrying out Kinematic Analysis, a number of NDOF motions should be specified
  - Kinematic Constraints vs. Driving Constraints

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$\mathbf{\Phi}^K(\mathbf{q}) \qquad\qquad\qquad \mathbf{\Phi}^D(\mathbf{q}, t)$$

Motion: Causes

- How can one set a mechanical system in motion?

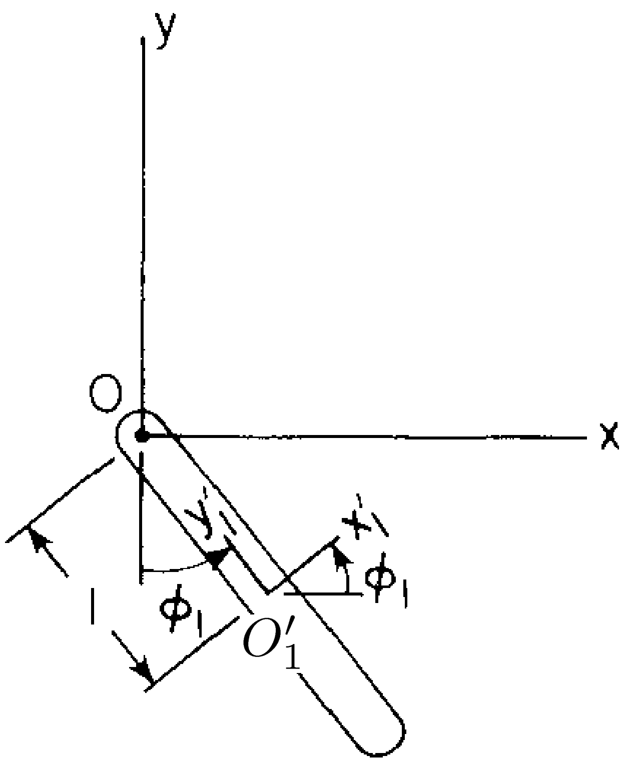
  - For a system with *ndof* degrees of freedom, specify NDOF additional driving constraints (one per degree of freedom) that uniquely determine **q**(t) as the solution of an algebraic problem (Kinematic Analysis)

  - Specify/Apply a set of forces acting upon the mechanism, in which case **q**(t) is found as the solution of a differential problem (Dynamic Analysis)

Ignore this for now…

Example 3.1.1

- A pin (revolute) joint present at point O
- Specify the set of constraints associated with this model
- A motion $\phi_1 = 4t^2$ is applied to the pendulum
- Use Cartesian coordinates



Kinematic Analysis Stages

- Position Analysis Stage
  - Challenging
- Velocity Analysis Stage
  - Simple
- Acceleration Analysis Stage
  - OK

- To take care of all these stages, ONE step is critical:
  - Write down the constraint equations associated with the joints present in your mechanism
- Once you have the constraints, the rest is boilerplate

Once you have the constraints…
(Going beyond the critical step)

- The three stages of Kinematics Analysis: <u>position</u> analysis, <u>velocity</u> analysis, and <u>acceleration</u> analysis they each follow *very* similar recipes for finding for each body of the mechanism its position, velocity and acceleration, respectively

- ALL STAGES RELY ON THE CONCEPT OF JACOBIAN MATRIX:
  - $\Phi_{\mathbf{q}}$ – the partial derivative of the constraints wrt the generalized coordinates

- ALL STAGES REQUIRE THE SOLUTION OF A SYSTEM OF EQUATIONS

$$\mathbf{\Phi_q}\,\mathbf{x} = \mathbf{b}$$

- WHAT IS *DIFFERENT* BETWEEN THE THREE STAGES IS THE EXPRESSION OF THE RIGHT-SIDE OF THE LINEAR EQUATION, “**b**”

The details…

- As we pointed out, it all boils down to this:
  - Step 1: Before anything, write down the constraint equations associated with your model
  - Step 2: For each stage, construct Φ_q and the specific $\mathbf{b}$, then solve for $\mathbf{x}$

- So how do you get the position configuration of the mechanism?
  - Kinematic Analysis key observation: The number of constraints (kinematic and driving) should be equal to the number of generalized coordinates
    - This is a prerequisite for Kinematic Analysis

$$\mathbf{\Phi}(\mathbf{q},t) = \begin{bmatrix} \mathbf{\Phi}^K(\mathbf{q}) \\ \mathbf{\Phi}^D(\mathbf{q},t) \end{bmatrix}_{nc\times 1} = \mathbf{0}$$

$$\mathbf{q} \in \mathbb{R}^{nc} \qquad \mathbf{\Phi} : \mathbb{R}^{nc+1} \rightarrow \mathbb{R}^{nc}$$

IMPORTANT: This is a nonlinear systems with *nc* equations and *nc* unknowns that you must solve to find **q**

Getting the Velocity and Acceleration of the Mechanism

- Previous slide taught us how to find the positions **q**
  - At each time step t_k, generalized coordinates $\mathbf{q}_k$ are the solution of a nonlinear system

- Take one time derivative of constraints Φ(q,t) to obtain the **velocity equation:**

$$\frac{d}{dt}\boldsymbol{\Phi}(\mathbf{q},t) = \mathbf{0} \qquad \Rightarrow \qquad \boldsymbol{\Phi}_{\mathbf{q}}\dot{\mathbf{q}} = -\boldsymbol{\Phi}_t$$

- Take yet one more time derivative to obtain the **acceleration equation:**

$$\ddot{\boldsymbol{\Phi}} = \frac{d^2}{dt^2}\boldsymbol{\Phi}(\mathbf{q},t) = \mathbf{0} \qquad \Rightarrow \qquad \boldsymbol{\Phi}_{\mathbf{q}}\ddot{\mathbf{q}} = -(\boldsymbol{\Phi}_{\mathbf{q}}\dot{\mathbf{q}})_{\mathbf{q}}\,\dot{\mathbf{q}} - 2\boldsymbol{\Phi}_{\mathbf{q}t}\,\dot{\mathbf{q}} - \boldsymbol{\Phi}_{tt}$$

- NOTE: Getting right-hand side of acceleration equation is tedious