ME451
Kinematics and Dynamics of Machine Systems

Review of Elements of Calculus – 2.5

Vel and Acc of a Point fixed in a Ref Frame – 2.6

September 14, 2010

ME451, UW-Madison

Before we get started…

- **NOTE**:
  - Next Tu and Th you'll go through two ADAMS tutorials
  - TA to email to lab where you'll meet for these two tutorials
- HW Assigned (due next Tu):
  - 2.4.4, 2.5.1,2.5.2, 2.5.3, 2.5.7
  - ADAMS component to be emailed to you by TA
- Last time:
  - Discussed about Recursive and Absolute Generalized Coordinates
    - Each has its own advantages/disadvantages
  - Brief linear algebra review
- Today:
  - Time derivatives of vectors and matrices
  - Computing the partial derivate of functions
  - Discuss chain rule for taking time derivatives
  - Computing the velocity and acceleration of a point attached to a moving rigid body

Derivatives of Functions

- GOAL: Understand how to
  - Take **time derivatives** of vectors and matrices
  - Take **partial derivatives** of functions with respect to its arguments
    - We will use a matrix-vector notation for computing these partial derivs.
    - Taking partial derivatives might be challenging in the beginning
    - It will be used a lot in this class

Taking time derivatives of a time dependent vector

- FRAMEWORK:

    - Vector **r** is represented as a function of time, and it has two components: x(t) and y(t):

$$\mathbf{r}(t) = \left[\begin{array}{c} x(t) \\ y(t) \end{array} \right]$$

    - Its components change, but the vector is represented in a **fixed** reference frame

- THEN:

$$\dot{\mathbf{r}}(t) = \left[\begin{array}{c} \dot{x}(t) \\ \dot{y}(t) \end{array} \right] \quad , \qquad \ddot{\mathbf{r}}(t) = \left[\begin{array}{c} \ddot{x}(t) \\ \ddot{y}(t) \end{array} \right] \quad , \qquad etc.$$

Time Derivatives, Vector Related Operations

- Assume that $\alpha \in \mathbb{R}$, $\mathbf{a} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^n$ depend on time. Then it can be proved that the following hold:

$$
\begin{aligned}
\frac{d}{dt}(\alpha\mathbf{a}) &= \frac{d\alpha}{dt}\mathbf{a} + \alpha\frac{d\mathbf{a}}{dt} = \dot{\alpha}\mathbf{a} + \alpha\dot{\mathbf{a}} \\
\frac{d}{dt}(\mathbf{a}+\mathbf{b}) &= \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{b}}{dt} = \dot{\mathbf{a}} + \dot{\mathbf{b}} \\
\frac{d}{dt}(\mathbf{a}^T\mathbf{b}) &= \frac{d\mathbf{a}^T}{dt}\mathbf{b} + \mathbf{a}^T\frac{d\mathbf{b}}{dt} = \dot{\mathbf{a}}^T\mathbf{b} + \mathbf{a}^T\dot{\mathbf{b}} \\
\mathbf{a}^T\mathbf{a} &= \text{const} \qquad \Rightarrow \qquad \mathbf{a}^T\dot{\mathbf{a}} = 0
\end{aligned}
$$

Taking time derivatives of MATRICES

- By <u>definition</u>, the time derivative of a matrix is obtained by taking the time derivative of each entry in the matrix

- A simple extension of what we've seen for vector derivatives

- Assume that $\alpha \in \mathbb{R}$, $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{m\times n}$, and $\mathbf{C} \in \mathbb{R}^{n\times p}$ depend on time. Then it can be proved that the following hold:

$$\begin{aligned}
\frac{d}{dt}(\alpha\mathbf{A}) &= \frac{d\alpha}{dt}\mathbf{A} + \alpha\frac{d\mathbf{A}}{dt} = \dot{\alpha}\mathbf{A} + \alpha\dot{\mathbf{A}} \\
\frac{d}{dt}(\mathbf{A}+\mathbf{B}) &= \frac{d\mathbf{A}}{dt} + \frac{d\mathbf{B}}{dt} = \dot{\mathbf{A}} + \dot{\mathbf{B}} \\
\frac{d}{dt}(\mathbf{A}\mathbf{C}) &= \frac{d\mathbf{A}}{dt}\mathbf{C} + \mathbf{A}\frac{d\mathbf{C}}{dt} = \dot{\mathbf{A}}\mathbf{C} + \mathbf{A}\dot{\mathbf{C}}
\end{aligned}$$

End Time Derivatives

…

Discuss Partial Derivatives

What's the story behind the concept of partial derivative?

- What's the meaning of a partial derivative?
  - It captures the "sensitivity" of a function quantity wrt a variable that the function depends upon
  - Shows how much the function changes when the variable changes a bit

- Simplest case of partial derivative: you have one function that depends on one variable:

$$f(x) = \ln x \quad , \qquad g(z) = sin(4z + \pi) \quad , \qquad etc.$$

- Then,

$$\frac{\partial f}{\partial x} = \frac{1}{x} \quad , \qquad \frac{\partial g}{\partial z} = 4\cos(4z + \pi) \quad , \qquad etc.$$

Partial Derivative, Two Variables

- Suppose you have one function but it depends on **two** variables, say x and y:

$$f(x,y) = \sin(x^2 + 3y^2)$$

- To simplify the notation, an array **q** is introduced:

$$\mathbf{q} = \left[\begin{array}{c} x \\ y \end{array} \right] \in \mathbb{R}^2$$

- With this, the partial derivative of f wrt **q** is defined as

$$[\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y}] \equiv f_{x,y} \equiv \frac{\partial f}{\partial \mathbf{q}} \equiv f_{\mathbf{q}} = [2x\cos(x^2+3y^2) \quad 6y\cos(x^2+3y^2)]$$

…and here is as good as it gets (vector function)

- You have a group of "m" functions that are gathered together in an array, and they depend on a collection of "n" variables:

$$f_1, f_2, \ldots, f_m \qquad \text{depend on} \qquad x_1, x_2, \ldots, x_n$$

- The array that collects all "m" functions is called **F**:

$$\mathbf{F}(x_1, x_2, \ldots, x_n) = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \cdots \\ f_m(x_1, x_2, \ldots, x_n) \end{bmatrix} \in \mathbb{R}^m$$

- The array that collects all "n" variables is called **q**:

$$\mathbf{q} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

Most general partial derivative (Vector Function, Cntd.)

- Then, in the most general case, we have **F**(**q**), and

$$\frac{\partial \mathbf{F}}{\partial \mathbf{q}} \equiv \mathbf{F_q} \equiv \begin{bmatrix} \frac{\partial f_1}{\partial \mathbf{q}} \\ \frac{\partial f_2}{\partial \mathbf{q}} \\ \cdots \\ \frac{\partial f_m}{\partial \mathbf{q}} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \in \mathbb{R}^{m \times n}$$

This is an m x n **matrix!**

- Example 2.5.2:

$$\mathbf{q} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \qquad \mathbf{r}^P = \begin{bmatrix} \cos\theta_1 + l\cos(\theta_1 + \theta_2) \\ \sin\theta_1 + l\sin(\theta_1 + \theta_2) \end{bmatrix} \qquad \mathbf{r}^P_{\mathbf{q}} = ?$$

A Word on Notation: Left and Right mean the same thing

- Let x, y, and ϕ be three generalized coordinates

- Define the function **r** of x, y, and ϕ as

$$\mathbf{r}(x, y, \phi) = \begin{bmatrix} x + 2l\cos\phi \\ y - 2l\sin\phi \end{bmatrix}$$

- Verbose notation

$$\mathbf{r}_{x,y,\phi} = \begin{bmatrix} \frac{\partial \mathbf{r}}{\partial x} & \frac{\partial \mathbf{r}}{\partial y} & \frac{\partial \mathbf{r}}{\partial \varphi} \end{bmatrix}$$

- Let x, y, and ϕ be three generalized coordinates, and define the array **q**

$$\mathbf{q} = \begin{bmatrix} x \\ y \\ \phi \end{bmatrix}$$

- Define the function **r** of **q**:

$$\mathbf{r}(\mathbf{q}) = \begin{bmatrix} x + 2l\cos\phi \\ y - 2l\sin\phi \end{bmatrix}$$

- Terse notation

$$\mathbf{r}_{\mathbf{q}} = \frac{\partial \mathbf{r}}{\partial \mathbf{q}}$$

Example

$$\mathbf{q} = \begin{bmatrix} x \\ y \\ \phi \end{bmatrix} \qquad \mathbf{r}(\mathbf{q}) = \begin{bmatrix} x + 2l\cos\phi \\ y - 2l\sin\phi \end{bmatrix} \qquad \mathbf{r_q} = \frac{\partial \mathbf{r}}{\partial \mathbf{q}} = ?$$

$$\mathbf{r_q} = \frac{\partial \mathbf{r}}{\partial \mathbf{q}} = \left[\frac{\partial \mathbf{r}}{\partial q_1} \quad \frac{\partial \mathbf{r}}{\partial q_2} \quad \frac{\partial \mathbf{r}}{\partial q_3}\right] = \left[\frac{\partial \mathbf{r}}{\partial x} \quad \frac{\partial \mathbf{r}}{\partial y} \quad \frac{\partial \mathbf{r}}{\partial \phi}\right]$$

$$\Downarrow$$

$$\mathbf{r_q} = \begin{bmatrix} 1 & 0 & -2l\sin\varphi \\ 0 & 1 & -2l\cos\varphi \end{bmatrix}$$

Another Example (builds on Example 2.4.1)

- Let

$$\mathbf{q} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

- Find the partial derivative of the position of P with respect to the array of generalized coordinates **q**

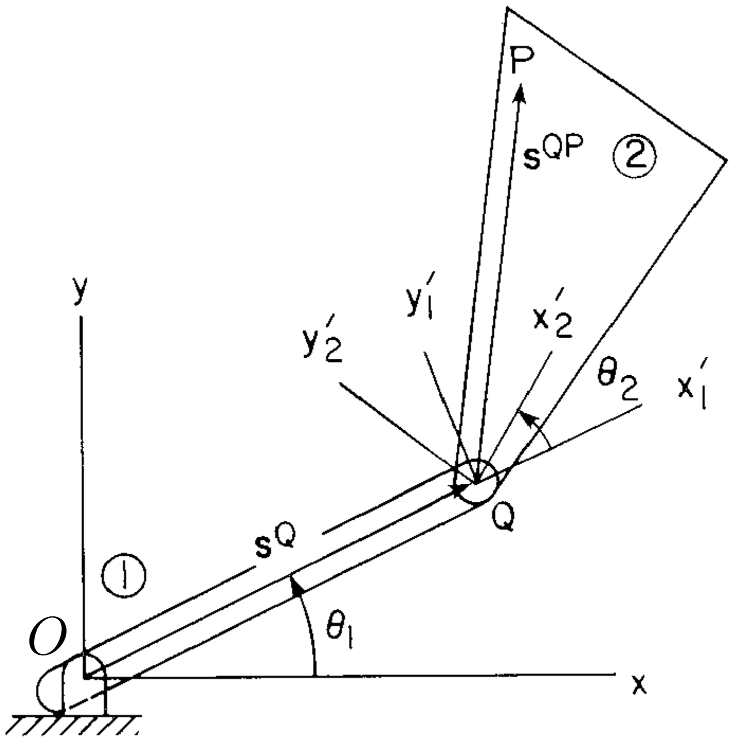

Figure 2.4.5 Two-body positioning mechanism.

$$s'^P = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\|\overrightarrow{OQ}\| = 5$$

$$\mathbf{r}^P_{\mathbf{q}} = \frac{\partial \mathbf{r}^P}{\partial \mathbf{q}} = [\frac{\partial \mathbf{r}^P}{\partial \theta_1} \quad \frac{\partial \mathbf{r}^P}{\partial \theta_2}] = ?$$

Partial Derivatives: Good to Remember…

- In the most general case, you start with “m” functions in “n” variables, and end with an (m x n) matrix of partial derivatives.
  - You start with a column vector of functions and then end up with a matrix

- Taking a partial derivative leads to a *higher dimension* quantity
    - Scalar Function – leads to row vector
    - Vector Function – leads to matrix
  - I call this the “accordion rule”

- In this class, taking partial derivatives can lead to one of the following:
    - A row vector
    - A full blown matrix
  - If you see something else chances are you made a mistake…

Done with Partial Derivatives

…

Moving on to Chain Rule of Differentiation

Scenario 1: ***Scalar*** Function

- f is a function of "n" variables: $q_1, \ldots, q_n$

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}$$

- However, each of these variables q_i in turn depends on a set of "k" other variables $x_1, \ldots, x_k$.

$$\mathbf{q} = \begin{bmatrix} q_1(x_1, \ldots, x_k) \\ \cdots \\ q_n(x_1, \ldots, x_k) \end{bmatrix} : \mathbb{R}^k \rightarrow \mathbb{R}^n$$

- The composition of f and **q** leads to a new function $\phi(\mathbf{x})$:

$$\phi(\mathbf{x}) = f \circ \mathbf{q} = f(\mathbf{q}(\mathbf{x})) : \mathbb{R}^k \longrightarrow \mathbb{R}$$

Chain Rule for a ***Scalar*** Function

- The question: how do you compute ϕ_x ?
  - Using our notation:

$$\phi = f \circ \mathbf{q} = f(\mathbf{q}(\mathbf{x})) \qquad \Rightarrow \qquad \phi_{\mathbf{x}} = \frac{\partial \phi}{\partial \mathbf{x}} = \; ??$$

- Chain Rule for scalar function:

$$\phi_{\mathbf{x}} = \frac{\partial \phi}{\partial \mathbf{x}} = \frac{\partial f}{\partial \mathbf{q}} \cdot \frac{\partial \mathbf{q}}{\partial \mathbf{x}} = f_{\mathbf{q}} \cdot \mathbf{q}_{\mathbf{x}}$$

Example

Assume that $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and a function ϕ of $\mathbf{y}$ is defined as: $\phi(\mathbf{y}) = 3y_1^2 + \sin y_2$.

In turn, $\mathbf{y}$ depends on a variable $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ as follows:

$$\mathbf{y} = \mathbf{y}(\mathbf{x}) = \begin{bmatrix} y_1(\mathbf{x}) \\ y_2(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} 2x_1 + \log x_2 + \sqrt{x_3} \\ (x_1 - x_2)^2 \end{bmatrix}$$

Now, since ϕ depends on $\mathbf{y}$ and $\mathbf{y}$ depends on $\mathbf{x}$, it means that ϕ depends on $\mathbf{x}$. Find the partial derivative of ϕ with respect to $\mathbf{x}$, that is,

$$\phi_{\mathbf{x}} = \frac{\partial \phi}{\partial \mathbf{x}} = [\frac{\partial \phi}{\partial x_1} \quad \frac{\partial \phi}{\partial x_2} \quad \frac{\partial \phi}{\partial x_3}] = ?$$

Scenario 2: ***Vector*** Function

- **F** is a function of "n" variables: $q_1, \ldots, q_n$

$$\mathbf{F} : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

- However, each of these variables q_i in turn depends on a set of "k" other variables $x_1, \ldots, x_k$.

$$\mathbf{q} = \begin{bmatrix} q_1(x_1, \ldots, x_k) \\ \cdots \\ q_n(x_1, \ldots, x_k) \end{bmatrix} : \mathbb{R}^k \rightarrow \mathbb{R}^n$$

- The composition of **F** and **q** leads to a new function $\Phi(\mathbf{x})$:

$$\mathbf{\Phi}(\mathbf{x}) = \mathbf{F} \circ \mathbf{q} = \mathbf{F}(\mathbf{q}(\mathbf{x})) : \mathbb{R}^k \rightarrow \mathbb{R}^m$$

Chain Rule for a ***Vector*** Function

- How do you compute the partial derivative of $\mathbf{\Phi}$?

$$\mathbf{\Phi} : \mathbb{R}^k \longrightarrow \mathbb{R}^m$$

$$\mathbf{\Phi} = \mathbf{\Phi}(\mathbf{q}(\mathbf{x})) \qquad \Rightarrow \qquad \mathbf{\Phi}_{\mathbf{x}} = \frac{\partial \mathbf{\Phi}}{\partial \mathbf{x}} = \; ??$$

- Chain rule for vector functions:

$$\mathbf{\Phi}_{\mathbf{x}} = \frac{\partial \mathbf{\Phi}}{\partial \mathbf{x}} = \frac{\partial \mathbf{F}}{\partial \mathbf{q}} \cdot \frac{\partial \mathbf{q}}{\partial \mathbf{x}}$$

Example

Assume that $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and a function $\mathbf{f}$ of $\mathbf{y}$ is defined as: $\mathbf{f}(\mathbf{y}) = \begin{bmatrix} 2y_1 + y_2^2 \\ y_1 y_2 \end{bmatrix}$.
In turn, $\mathbf{y}$ depends on a variable $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ as follows:

$$\mathbf{y} = \mathbf{y}(\mathbf{x}) = \begin{bmatrix} y_1(\mathbf{x}) \\ y_2(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} x_1 x_2 \\ x_1^2 - x_2 \end{bmatrix}$$

Now, since $\mathbf{f}$ depends on $\mathbf{y}$ and $\mathbf{y}$ depends on $\mathbf{x}$, it means that $\mathbf{f}$ depends on $\mathbf{x}$.
Find the partial derivative of $\mathbf{f}$ with respect to $\mathbf{x}$, that is,

$$\mathbf{f}_{\mathbf{x}} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} = [\frac{\partial \mathbf{f}}{\partial x_1} \quad \frac{\partial \mathbf{f}}{\partial x_2}] = ?$$

Scenario 3: Function of Two Vectors

- **F** is a vector function of 2 vector variables **q** and **p** :

$$\mathbf{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

- Both **q** and **p** in turn depend on a set of "k" other variables **x**=$[x_1, \ldots, x_k]^T$:

$$\mathbf{q} = \mathbf{q}(x_1, \ldots, x_k) : \mathbb{R}^k \rightarrow \mathbb{R}^{n_1}$$

$$\mathbf{p} = \mathbf{p}(x_1, \ldots, x_k) : \mathbb{R}^k \rightarrow \mathbb{R}^{n_2}$$

$$n = n_1 + n_2$$

- A new function Φ(**x**) is defined as:

$$\mathbf{\Phi}(\mathbf{x}) = \mathbf{F}(\mathbf{q}(\mathbf{x}), \mathbf{p}(\mathbf{x})) : \mathbb{R}^k \rightarrow \mathbb{R}^m$$

The Chain Rule

- How do you compute the partial derivative of $\mathbf{\Phi}$ with respect to **x** ?

$$\mathbf{\Phi} : \mathbb{R}^k \longrightarrow \mathbb{R}^m$$

$$\mathbf{\Phi} = \mathbf{\Phi}(\mathbf{x}) \qquad \Rightarrow \qquad \mathbf{\Phi}_{\mathbf{x}} = \frac{\partial \mathbf{\Phi}}{\partial \mathbf{x}} = \; ??$$

- Chain rule for function of two vectors:

$$\mathbf{\Phi}_{\mathbf{x}} = \frac{\partial \mathbf{\Phi}}{\partial \mathbf{x}} = \frac{\partial \mathbf{F}}{\partial \mathbf{q}} \cdot \frac{\partial \mathbf{q}}{\partial \mathbf{x}} + \frac{\partial \mathbf{F}}{\partial \mathbf{p}} \cdot \frac{\partial \mathbf{p}}{\partial \mathbf{x}} = \mathbf{F}_{\mathbf{q}} \cdot \mathbf{q}_{\mathbf{x}} + \mathbf{F}_{\mathbf{p}} \cdot \mathbf{p}_{\mathbf{x}}.$$

Example:

Assume that $\mathbf{q} = \mathbf{q}(\mathbf{x}) \in \mathbb{R}^3$, and $\mathbf{p} = \mathbf{p}(\mathbf{x}) \in \mathbb{R}^3$. Show that:

$$\frac{\partial(\mathbf{q}^T \mathbf{p})}{\partial \mathbf{x}} = \mathbf{q}^T \mathbf{p}_{\mathbf{x}} + \mathbf{p}^T \mathbf{q}_{\mathbf{x}}$$

Scenario 4: Time Derivatives

- In the previous slides we talked about functions f of y, while y in turn depended on yet another variable x

- The relevant case is when the variable x is actually time, t
  - This scenario is super common in 451:
    - You have a function that depends on the generalized coordinates **q**, and in turn the generalized coordinates are functions of time (they change in time, since we are talking about kinematics/dynamics here…)

  - Case 1: scalar function that depends on an array of m generalized coordinates that in turn depend on time

$$\Phi = \Phi(\mathbf{q}(t)) \in \mathbb{R}$$

  - Case 2: vector function (of dimension n) that depends on an array of m generalized coordinates that in turn depend on time

$$\boldsymbol{\Phi} = \boldsymbol{\Phi}(\mathbf{q}(t)) \in \mathbb{R}^n$$

A Special Case: Time Derivatives (Cntd)

- Of interest if finding the time derivative of Φ and $\mathbf{\Phi}$

- Apply the chain rule, the scalar function Φ case first:

$$\dot{\Phi} = \frac{d\Phi}{dt} = \frac{d\Phi(\mathbf{q}(t))}{dt} = \frac{\partial \Phi}{\partial \mathbf{q}} \cdot \frac{d\mathbf{q}}{dt} = \Phi_{\mathbf{q}}\dot{\mathbf{q}} \in \mathbb{R}$$

- For the vector function case, applying the chain rule leads to the same formula, only the size of the result is different…

$$\dot{\mathbf{\Phi}} = \frac{d\mathbf{\Phi}}{dt} = \frac{d\mathbf{\Phi}(\mathbf{q}(t))}{dt} = \frac{\partial \mathbf{\Phi}}{\partial \mathbf{q}} \cdot \frac{d\mathbf{q}}{dt} = \mathbf{\Phi}_{\mathbf{q}}\dot{\mathbf{q}} \in \mathbb{R}^n$$

Example, Scalar Function Φ

- Assume a set of generalized coordinates is defined through array **q**. Also, a scalar function Φ of **q** is provided:

$$\mathbf{q}(t) = \begin{bmatrix} x(t) \\ y(t) \\ \theta(t) \end{bmatrix}$$

$$\Phi(\mathbf{q}) = 3x(t) + 2L\sin\theta(t)$$

- Find time derivative of Φ

$$\dot{\Phi} = ?$$

Example, Vector Function Φ

- Assume a set of generalized coordinates is defined through array **q**. Also, a vector function Φ of **q** is provided:

$$\mathbf{q}(t) = \begin{bmatrix} x(t) \\ y(t) \\ \theta(t) \end{bmatrix}$$

$$\mathbf{\Phi}(\mathbf{q}) = \begin{bmatrix} 3x(t) & + & 2L\sin\theta(t) \\ y(t) & - & 2L\cos\theta(t) \end{bmatrix}$$

- Find time derivative of $\mathbf{\Phi}$

$$\dot{\mathbf{\Phi}} = ?$$

Useful Formulas

- A couple of useful formulas, some of them you had to derive as part of the HW

$$\frac{\partial(\mathbf{g}^T\mathbf{p})}{\partial\mathbf{q}} = \mathbf{g}^T\mathbf{p_q} + \mathbf{p}^T\mathbf{g_q}$$

$$\frac{\partial}{\partial\mathbf{q}}(\mathbf{Cq}) = \mathbf{C}$$

$$\frac{\partial}{\partial\mathbf{x}}(\mathbf{x}^T\mathbf{Cy}) = \mathbf{y}^T\mathbf{C}^T$$

$$\frac{d}{dt}(\mathbf{p}^T\mathbf{Cq}) = \dot{\mathbf{p}}^T\mathbf{Cq} + \mathbf{p}^T\mathbf{C}\dot{\mathbf{q}}$$

Assumptions:
$\mathbf{g} = \mathbf{g}(\mathbf{q})$
$\mathbf{p} = \mathbf{p}(\mathbf{q})$
$\mathbf{C}$ - constant matrix
$\mathbf{y}$ doesn't depend on $\mathbf{x}$

The dimensions of the vectors and matrix above such that all the operations listed can be carried out.

Example

- Derive the last equality on previous slide
- Can you expand that equation further?

Assumptions:
$\mathbf{p} = \mathbf{p}(\mathbf{q})$
$\mathbf{C}$ - constant matrix

$$\frac{d}{dt}(\mathbf{p}^T \mathbf{C} \mathbf{q}) = \dot{\mathbf{p}}^T \mathbf{C} \mathbf{q} + \mathbf{p}^T \mathbf{C} \dot{\mathbf{q}}$$