ME451
Kinematics and Dynamics of Machine Systems

Review of Matrix Algebra – 2.2
Review of Elements of Calculus – 2.5
Vel. and Acc. of a point fixed in a Ref Frame – 2.6

September 9, 2010

Dan Negrut
University of Wisconsin, Madison

Before we get started…

- Due next week:
  - ADAMS assignment due on Wd (email solution directly to *TA*)
    - ADAMS questions: please contact TA directly (jcmadsen@wisc.edu)
  - Problems: 2.2.5, 2.2.8. 2.2.10 out of Haug's book (due Tuesday) (http://sbel.wisc.edu/Courses/ME451/2010/bookHaugPointers.htm)

- Last time:
  - Covered Geometric Vectors & operations with them
  - Justified the need for Reference Frames
  - Introduced algebraic representation of a vector & related operations
  - Rotation Matrix (for switching from one RF to another RF)

- Today:
  - Dealing with bodies that are offset (unfinished business)
  - Discuss concept of "generalized coordinates"
  - Quick review of matrix/vector algebra

Translation + Rotation

- What we've covered so far deals with the case when you are interested in finding the representing the location of a point P when you change the RF, yet the new and old reference frames share the same origin

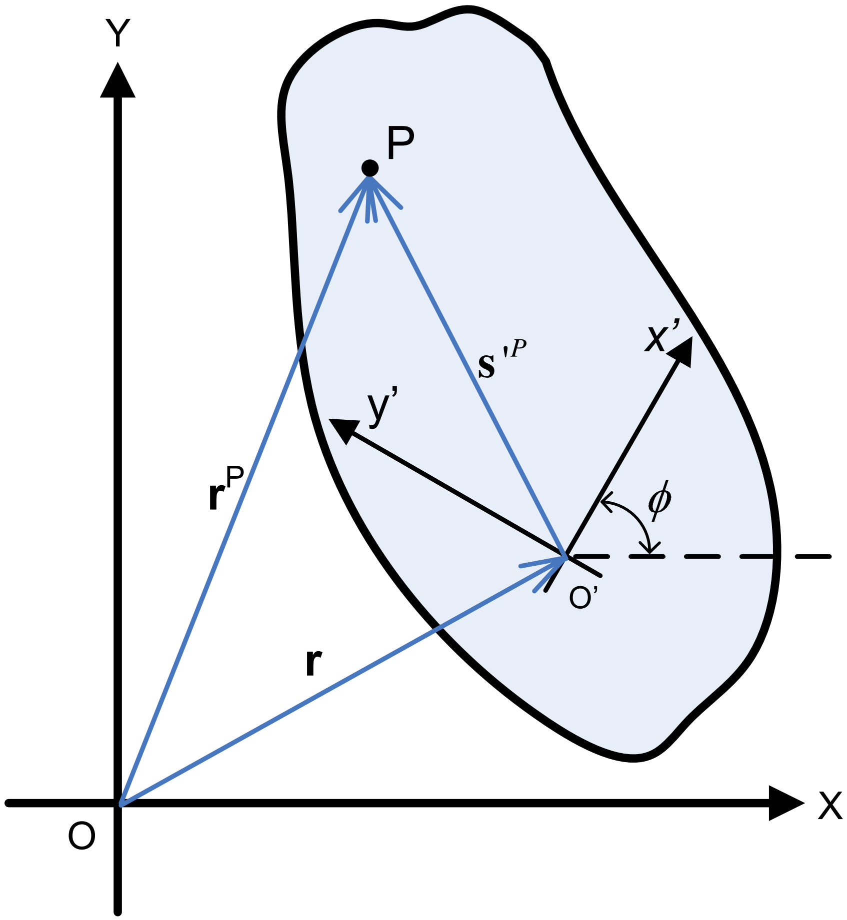
- What if they don't share the same origin? How would you represent the position of the point P in this new reference frame?

- Geometric Representation:

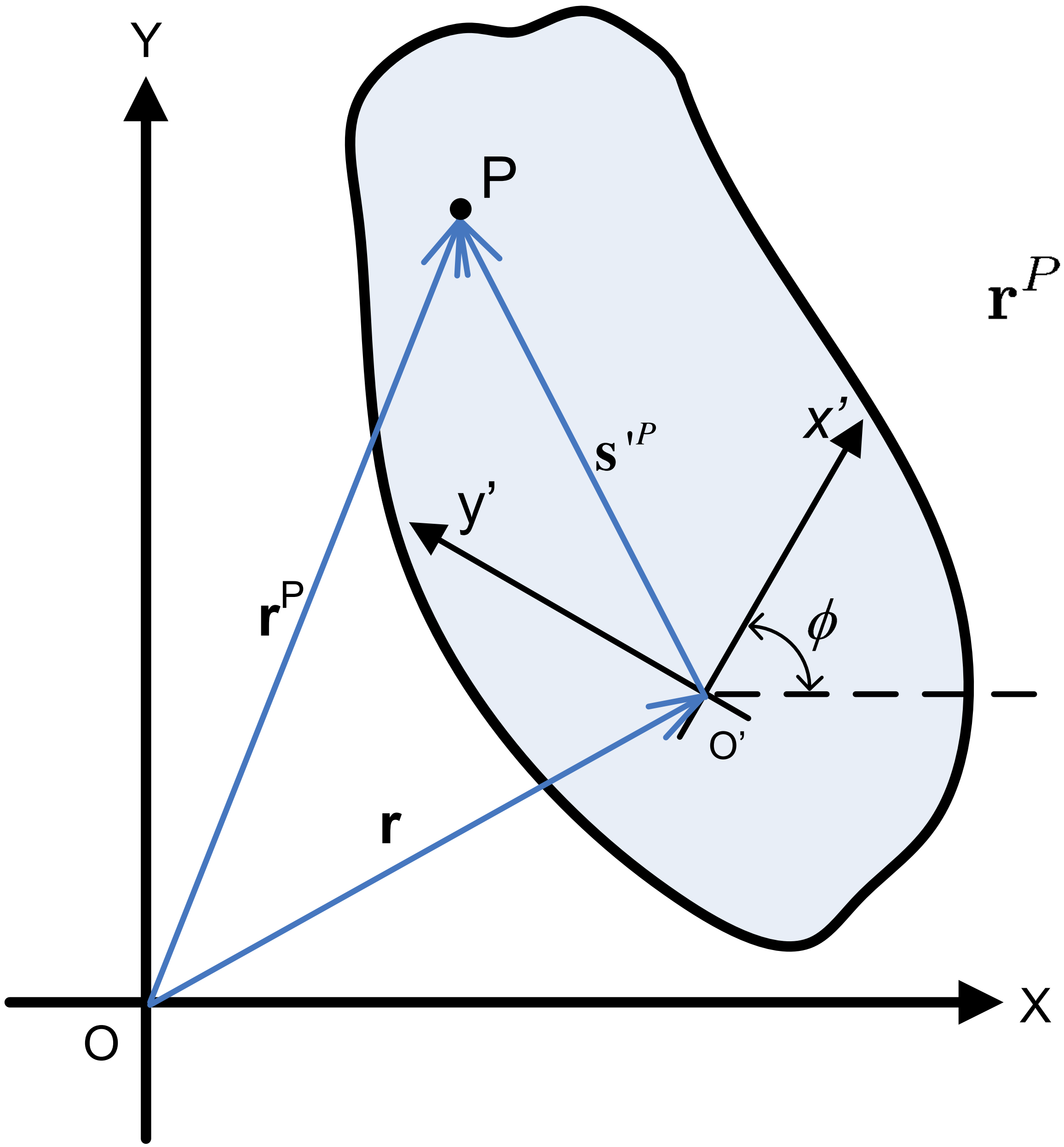
$$\overrightarrow{OP} = \overrightarrow{OO'} + \overrightarrow{O'P}$$

- Algebraic Representation:

$$\mathbf{r}^P = \mathbf{r} + \mathbf{s}^P = \mathbf{r} + \mathbf{A}\mathbf{s}'^P$$



Important Slide



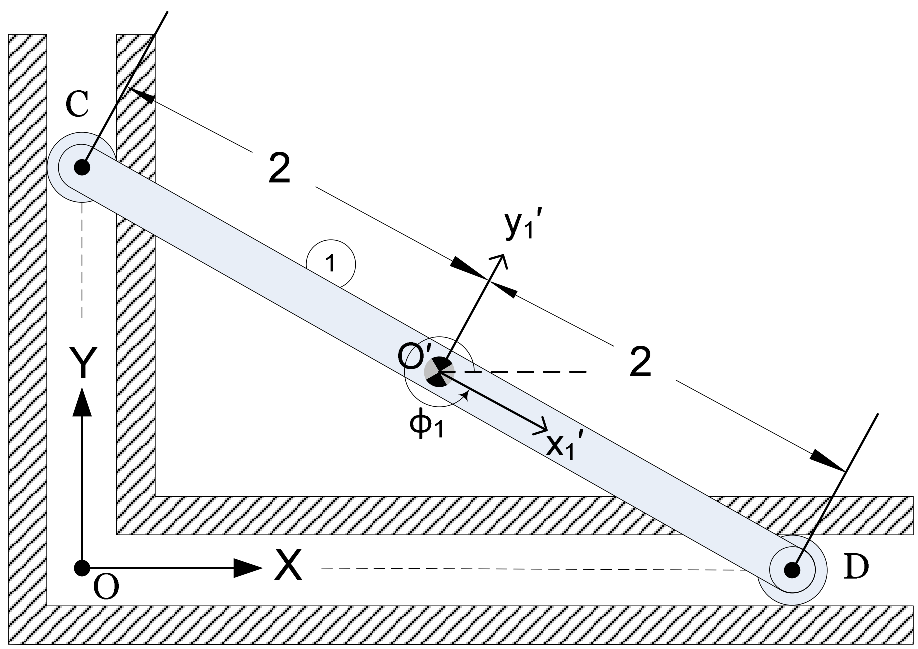
$$\mathbf{r}^P = \mathbf{r} + \mathbf{s}^P = \mathbf{r} + \mathbf{A}\mathbf{s}'^P$$

$$\mathbf{s} = \mathbf{A}\mathbf{s}'$$

$$\mathbf{A} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

Example

- The location of point O' in the OXY global RF is $[x,y]^T$. The orientation of the bar is described by the angle ϕ_1. Find the location of C and D expressed in the global reference frame as functions of x, y, and ϕ_1.



Absolute (Cartesian) Generalized Coordinates

vs.

Relative Generalized Coordinates

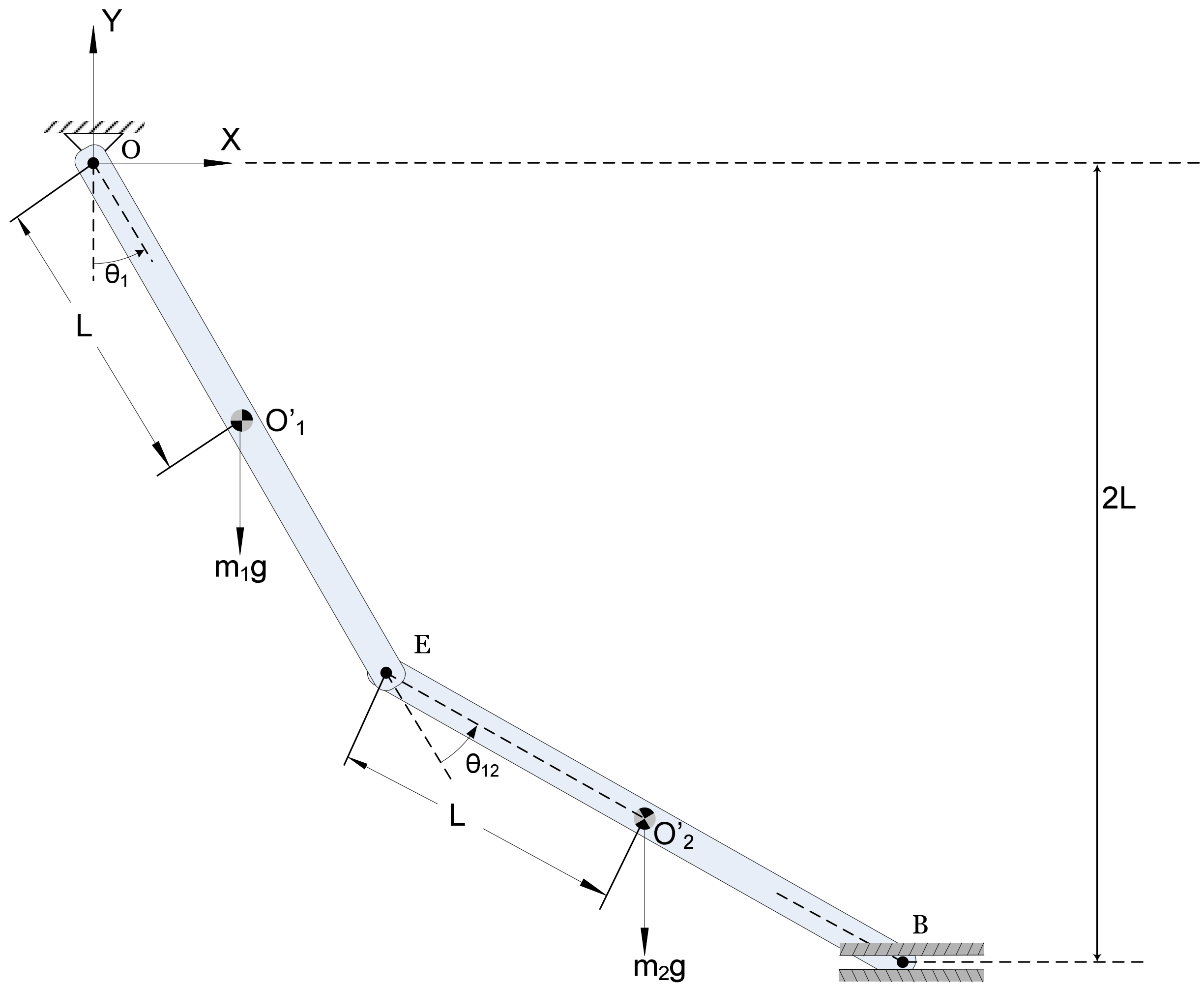
Generalized Coordinates

[General Comments]

- Generalized coordinates: What are they?
  - A set of quantities (variables) that allow you to uniquely determine the state of the mechanism
    - You need to know the location of each body
    - You need to know the orientation of each body
  - The quantities (variables) are bound to change in time since our mechanism moves
    - In other words, the generalized coordinates are functions of time
  - The rate at each the generalized coordinates change is capture by the set of generalized velocities
    - Most often, obtained as the straight time derivative of the generalized coordinates
    - Sometimes this is not the case though
      - Example: in 3D Kinematics, there is generalized coordinate whose time derivative is the angular velocity
- Important remark: there are multiple ways of choose the set of generalized coordinates that describe the state of your mechanism
  - We'll briefly look at two choices next

Example (RGC)

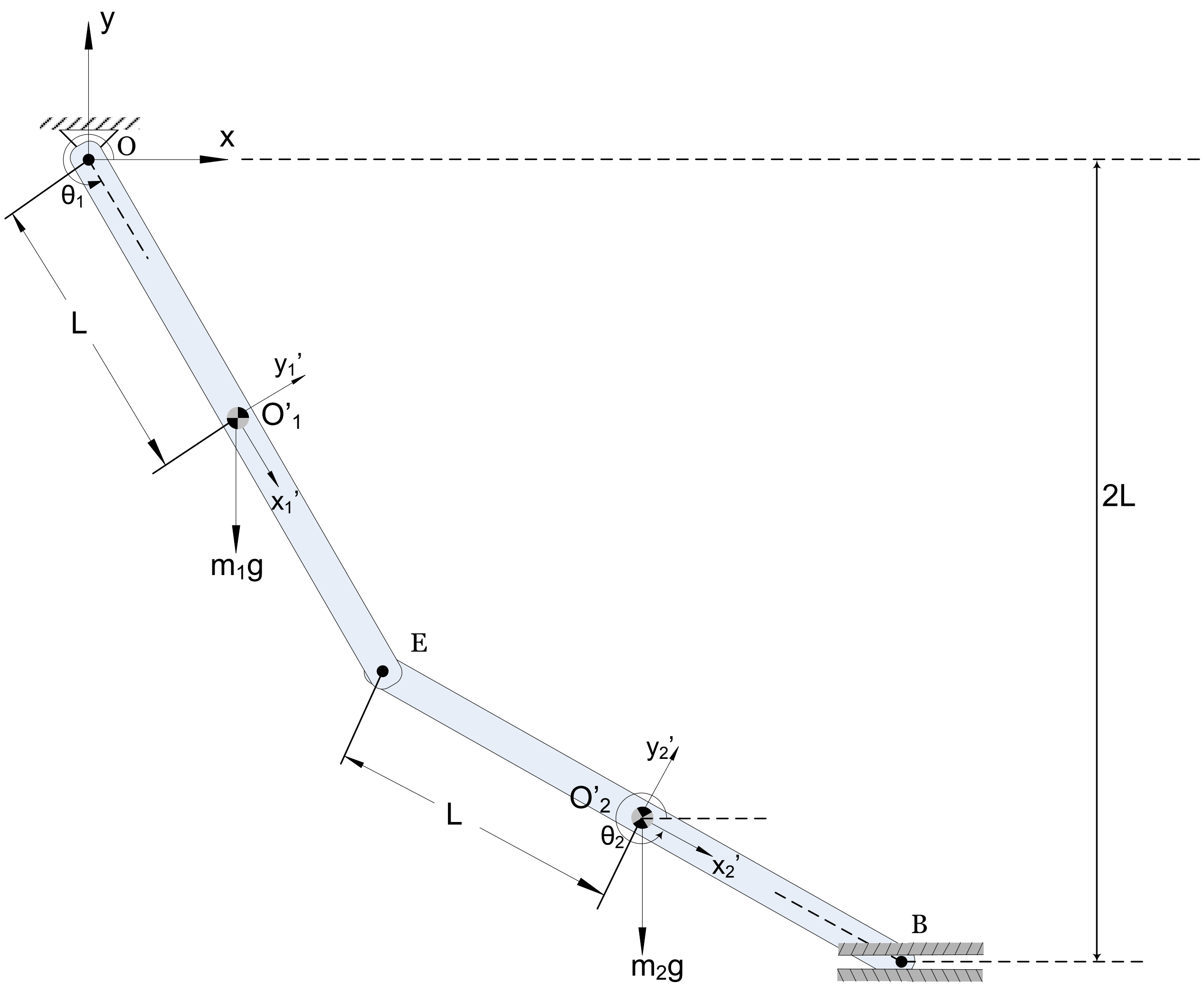
- Use the array **q** of generalized coordinates to locate the point B in the GRF (Global Reference Frame)



$$\mathbf{q} = \begin{bmatrix} \theta_1 \\ \theta_{12} \end{bmatrix}$$

Example (AGC)

- Use array **q** of generalized coordinates to locate the point B in the GRF (Global Reference Frame)



$$\mathbf{q} = \begin{bmatrix} x_2 \\ y_2 \\ \theta_2 \end{bmatrix}$$

Relative vs. Absolute Generalized Coordinates

- A consequential question:
  - Where was it easier to come up with position of point B?

- First Approach (Example RGC) – relies on relative coordinates:
  - Angle θ_1 uniquely specified both position and orientation of body 1
  - Angle θ_{12} uniquely specified the position and orientation of body 2 with respect to body 1
    - To locate B wrt global RF, first I position it with respect to body 1 (drawing on θ_{12}), and then locate the latter wrt global RF (based on θ_1)
    - Note that if there were 100 bodies, I would have to position wrt to body 99, which then I locate wrt body 98, …, and finally position wrt global RF

Relative vs. Absolute Generalized Coordinates (Cntd)

- Second Approach (Example AGC) – relies on absolute (and Cartesian) generalized coordinates:
  - x_1, y_1, θ_1 position and orient body 1 wrt GRF (global RF)
  - x_2, y_2, θ_2 position and orient body 2 wrt GRF (global RF)
    - To express the location of B is then very straightforward, use only x_2, y_2, θ_2 and local information (local position of B in body 2)
  - For AGC, you handle many generalized coordinates
    - 3 for each body in the system (six for this example)

Relative vs. Absolute Generalized Coordinates (Cntd)

- Conclusion for AGC and RGC:
  - There is no free lunch:
    - AGC: easy to express locations but many GCs
    - RGC: few GCs but cumbersome process of locating B
  - Personally, I prefer AGC, the math is simple…
- RGC common in robotics and molecular dynamics
- AGC common in multibody dynamics

Example 2.4.3: Slider Crank

- Based on information provided in figure (b), derive the position vector associated with point P (that is, find position of point P in the global reference frame OXY)

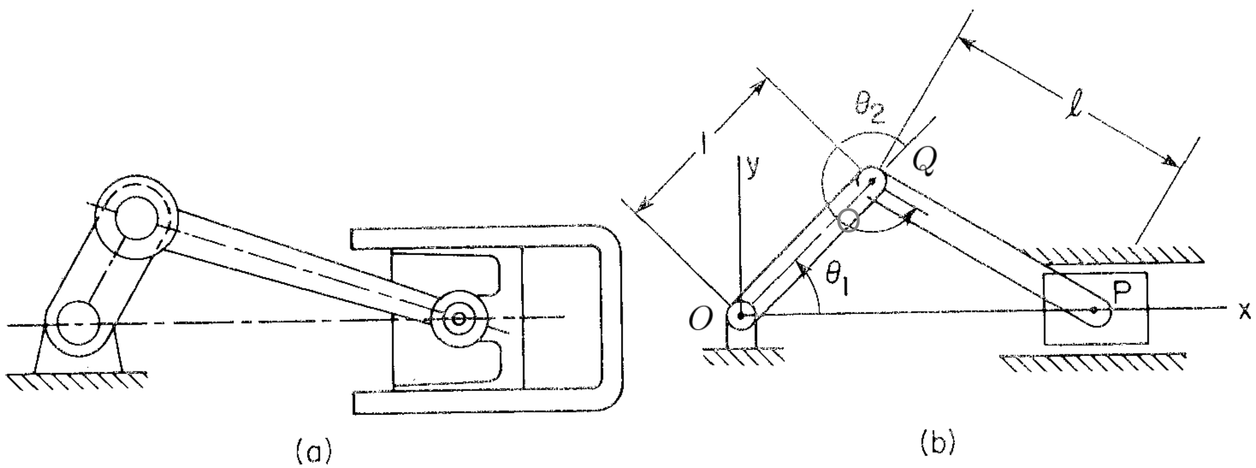


Figure 2.4.6 Slider–crank mechanism. (a) Physical system. (b) Kinematic model.

Begin Matrix Review

Notation Conventions

- A bold upper case letter denotes matrices
  - Example: **A**, **B**, etc.
- A bold lower case letter denotes a vector
  - Example: **v**, **s**, etc.
- A letter in italics format denotes a scalar quantity
  - Example: a, b_1

Matrix Review

- What is a matrix?

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{\alpha}_1^T \\ \boldsymbol{\alpha}_2^T \\ \dots \\ \boldsymbol{\alpha}_m^T \end{bmatrix}$$

- Matrix addition:

$$\mathbf{A} = [a_{ij}], \qquad 1 \le i \le m, \qquad 1 \le j \le n$$

$$\mathbf{B} = [b_{ij}], \qquad 1 \le i \le m, \qquad 1 \le j \le n$$

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = [c_{ij}], \qquad c_{ij} = a_{ij} + b_{ij}$$

  - Addition is commutative

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

Matrix Multiplication

- Recall dimension constraints on matrices so that they can be multiplied:
  - # of columns of first matrix is equal to # of rows of second matrix

$$\mathbf{A} = [a_{ij}], \qquad \mathbf{A} \in \mathbb{R}^{m*n}$$

$$\mathbf{C} = [c_{ij}], \qquad \mathbf{C} \in \mathbb{R}^{n*p}$$

$$\mathbf{D} = \mathbf{A} \cdot \mathbf{C} = [d_{ij}], \qquad \mathbf{D} \in \mathbb{R}^{m*p}$$

$$d_{ij} = \sum_{k=1}^{n} a_{ik} c_{kj}$$

- Matrix multiplication operation is not commutative

- Distributivity of matrix multiplication with respect to matrix addition:

$$(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$$

Matrix-Vector Multiplication

- A column-wise perspective on matrix-vector multiplication

$$\mathbf{A}\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix} = \sum_{i=1}^{n} \mathbf{a}_i v_i$$

  - Example:

$$\mathbf{A}\mathbf{v} = \begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 3 & 1 & 1 \\ -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} \cdot (1) + \begin{bmatrix} 4 \\ 3 \\ 0 \\ 1 \end{bmatrix} \cdot (2) + \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix} \cdot (-1) + \begin{bmatrix} 0 \\ 1 \\ -1 \\ -2 \end{bmatrix} \cdot (1) = \begin{bmatrix} 7 \\ 8 \\ -3 \\ 1 \end{bmatrix}$$

- A row-wise perspective on matrix-vector multiplication:

$$\mathbf{A}\mathbf{v} = \begin{bmatrix} \boldsymbol{\alpha}_1^T \\ \boldsymbol{\alpha}_2^T \\ \dots \\ \boldsymbol{\alpha}_m^T \end{bmatrix} \mathbf{v} = \begin{bmatrix} \boldsymbol{\alpha}_1^T \mathbf{v} \\ \boldsymbol{\alpha}_2^T \mathbf{v} \\ \dots \\ \boldsymbol{\alpha}_m^T \mathbf{v} \end{bmatrix}$$

Orthogonal & Orthonormal Matrices

- Definition (**Q**, orthogonal matrix): a square matrix **Q** is orthogonal if the product $\mathbf{Q}^T\mathbf{Q}$ is a diagonal matrix

- Matrix **Q** is called orthonormal if it's orthogonal and also $\mathbf{Q}^T\mathbf{Q}=\mathbf{I}_n$
  - Note that people in general don't make a distinction between an orthogonal and orthonormal matrix

- Note that if **Q** is an orthonormal matrix, then $\mathbf{Q}^{-1}=\mathbf{Q}^T$

- Example, orthonormal matrix:

$$\mathbf{Q} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Exercise

- Prove that the orientation **A** matrix is orthonormal

$$\mathbf{A} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

Remark:

On the Columns of an Orthonormal Matrix

- Assume **Q** is an orthonormal matrix

$$\mathbf{Q} \in \mathbb{R}^{n \times n} \qquad \mathbf{Q} = [\mathbf{q}_1, ..., \mathbf{q}_n] \quad \leftarrow \quad \text{orthonormal}$$

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I}_{n \times n} \qquad \Rightarrow \qquad \begin{bmatrix} \mathbf{q}_1^T \\ .. \\ \mathbf{q}_n^T \end{bmatrix} [\mathbf{q}_1, ..., \mathbf{q}_n] = \begin{bmatrix} \mathbf{q}_1^T\mathbf{q}_1 & .. & \mathbf{q}_1^T\mathbf{q}_n \\ .. & .. & .. \\ \mathbf{q}_n^T\mathbf{q}_1 & .. & \mathbf{q}_n^T\mathbf{q}_n \end{bmatrix} = \mathbf{I}_{n \times n}$$

$$\Downarrow$$

$$\mathbf{q}_i^T \cdot \mathbf{q}_j = \delta_{ij} = \begin{cases} 1 & if \quad i = j \\ 0 & if \quad i \neq j \end{cases}$$

- In other words, the columns (and the rows) of an orthonormal matrix have unit norm and are mutually perpendicular to each other

Matrix Review [Cntd.]

- Scaling of a matrix by a real number: scale each entry of the matrix

$$\alpha \cdot \mathbf{A} = \alpha \cdot [a_{ij}] = [\alpha \cdot a_{ij}]$$

  - Example:

$$(1.5) \cdot \begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 3 & 1 & 1 \\ -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 1.5 & 6 & 3 & 0 \\ 3 & 4.5 & 1.5 & 1.5 \\ -1.5 & 0 & 1.5 & -1.5 \\ 0 & 1.5 & -1.5 & -3 \end{bmatrix}$$

- Transpose of a matrix **A** dimension m$\times$n: a matrix **B**=**A**$^{\mathbf{T}}$ of dimension n$\times$m whose (i,j) entry is the (j,i) entry of original matrix **A**

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 3 & 1 & 1 \\ -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & -2 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 4 & 3 & 0 & 1 \\ 2 & 1 & 1 & -1 \\ 0 & 1 & -1 & -2 \end{bmatrix}$$

Linear Independence of Vectors

- Definition: linear independence of a set of m vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_m$:

$$\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{R}^n$$

- The vectors are linearly independent if the following condition holds

$$\alpha_1 \mathbf{v}_1 + \ldots + \alpha_m \mathbf{v}_m = \mathbf{0}_n \quad \Rightarrow \quad \alpha_1 = \ldots = \alpha_m = 0$$

- If a set of vectors are not linearly independent, they are called dependent
  - Example:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ -3 \\ -6 \end{bmatrix}$$

  - Note that $\mathbf{v}_1 - 2\mathbf{v}_2 - \mathbf{v}_3 = 0$

Matrix Rank

- Row rank of a matrix
  - Largest number of rows of the matrix that are linearly independent
  - A matrix is said to have full row rank if the rank of the matrix is equal to the number of rows of that matrix

- Column rank of a matrix
  - Largest number of columns of the matrix that are linearly independent

- NOTE: for each matrix, the row rank and column rank are the same
  - This number is simply called the rank of the matrix
  - It follows that

$$\text{rank}(C) = \text{rank}(C^T)$$

Matrix Rank, Example

- What is the row rank of the matrix **J**?

$$\mathbf{J} = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 4 & -2 & -2 & 1 \\ 0 & -4 & 0 & 1 \end{bmatrix}$$

- What is the rank of **J**?

Matrix Review [Cntd.]

- Symmetric matrix: a square matrix **A** for which $\mathbf{A}=\mathbf{A}^T$
- Skew-symmetric matrix: a square matrix **B** for which $\mathbf{B}=-\mathbf{B}^T$
- Examples:

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 3 \\ -1 & 3 & 4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 4 \\ -2 & -4 & 0 \end{bmatrix}$$

- Singular matrix: square matrix whose determinant is zero

$$\det(\mathbf{A}) = 0, \qquad \mathbf{A} \in \mathbb{R}^{n\times n}$$

- Inverse of a square matrix **A**: a matrix of the same dimension, called $\mathbf{A}^{-1}$, that satisfies the following:

$$\mathbf{A}^{-1}\cdot\mathbf{A} = \mathbf{A}\cdot\mathbf{A}^{-1} = \mathbf{I}_n, \qquad \mathbf{A} \in \mathbb{R}^{n\times n}$$

Singular vs. Nonsingular Matrices

- Let **A** be a square matrix of dimension *n*. The following are equivalent:

  - $\mathbf{Ax} = \mathbf{b}$ has a unique solution for any $\mathbf{b} \in \mathbb{R}^n$.
  - $\mathbf{Ax} = \mathbf{b}$ has a solution for any $\mathbf{b} \in \mathbb{R}^n$.
  - $\mathbf{Ax} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$.
  - $\mathbf{A}^{-1}$ exists.
  - $\text{Determinant}(\mathbf{A}) \neq 0$.
  - $\text{rank}(\mathbf{A}) = n$.

Other Useful Formulas

[Pretty straightforward to prove true]

- If **A** and **B** are invertible, their product is invertible and

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

- Also,

$$\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$$

- For any two matrices **A** and **B** that can be multiplied

$$(\mathbf{AB})^{T} = \mathbf{B}^{T}\mathbf{A}^{T}$$

- For any three matrices **A**, **B**, and **C** that can be multiplied

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$$