ME451
Kinematics and Dynamics of Machine Systems

Review of Linear Algebra
2.1 through 2.4
Tu, Sept. 07
Before we get started...

- Last time:
  - Syllabus
  - Quick overview of course
  - Starting discussion about vectors, their geometric representation

- HW Assigned:
  - ADAMS assignment, will be emailed to you today
  - Problems: 2.2.5, 2.2.8, 2.2.10
  - Due in one week
Geometric Entities: Their Relevance

- Kinematics (and later Dynamics) of systems of rigid bodies:
  - Requires the ability to describe the position, velocity, and acceleration of each rigid body in the system as functions of time

- In the Euclidian 2D space, geometric vectors and 2X2 orthonormal matrices are extensively used to this end

  - Geometric vectors - used to locate points on a body or the center of mass of a rigid body

  - 2X2 orthonormal matrices - used to describe the orientation of a body
Geometric Vectors

● What is a “Geometric Vector”?
  ● A quantity that has three attributes:
    ● A direction (given by the blue line)
    ● A sense (from O to P)
    ● A magnitude, ||OP||
  ● Note that all geometric vectors are defined in relation to an origin O

● IMPORTANT:
  ● Geometric vectors are entities that are independent of any reference frame

● ME451 deals planar kinematics and dynamics
  ● We assume that all the vectors are defined in the 2D Euclidian space
  ● A basis for the Euclidian space is any collection of two independent vectors
Geometric Vectors: Operations

- What geometric vectors operations are defined out there?
  - Scaling by a scalar $\alpha$
  - Addition of geometric vectors (the parallelogram rule)
  - Multiplication of two geometric vectors
    - The inner product rule (leads to a number)
    - The outer product rule (leads to a vector)
  - One can measure the angle $\theta$ between two geometric vectors
- A review these definitions follows over the next couple of slides
G. Vector Operation: Scaling by $\alpha$

- By definition, scaling one geometric vector $\vec{a}$ by a scalar $\alpha \neq 0$ leads to a new vector $\vec{b} \equiv \alpha \vec{a}$ that has the following three attributes:
  - $\vec{b}$ has the same direction as the vector $\vec{a}$
  - $\vec{b}$ has the sense of $\vec{a}$ if $\alpha > 0$, and opposite sense if $\alpha < 0$
  - The magnitude of $\vec{b}$ is $b = |\alpha|a$

- Note that if $\alpha = 0$, then $\vec{b}$ is the null vector.
G. Vector Operation:
Addition of Two G. Vectors

- Sum of two vectors (definition)
  - Obtained by the parallelogram rule
- Operation is commutative
- Easy to visualize, pretty messy to summarize in an analytical fashion:

\[
c = \sqrt{||OR||^2 + ||RC||^2} = \sqrt{(a + b \cos \theta)^2 + b^2 \sin^2 \theta}
\]

\[
\alpha = \tan^{-1} \frac{b \sin \theta}{a + b \cos \theta}
\]
**G. Vector Operation:**

**Inner Product of Two G. Vectors**

- The product between the magnitude of the first geometric vector and the projection of the second vector onto the first vector

\[ \vec{a} \cdot \vec{b} = ||\vec{a}|| \cdot ||\vec{b}|| \cdot \cos(\vec{a}, \vec{b}) \]

- Note that operation is commutative

- Don’t call this the “dot product” of the two vectors
  - This name is saved for algebraic vectors
Regarding the angle between two vectors, note that

\[ \theta(\vec{a}, \vec{b}) \neq \theta(\vec{b}, \vec{a}) \]
\[ \theta(\vec{a}, \vec{b}) = 2\pi - \theta(\vec{b}, \vec{a}) \]

Important: Angles are positive \textbf{counterclockwise}.

This is why when measuring the angle between two vectors it’s important what the first (start) vector is.
Combining Basic G. Vector Operations

- **P1** – The sum of geometric vectors is associative
  \[
  \mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}
  \]

- **P2** – Multiplication with a scalar is distributive with respect to the sum:
  \[
  k \cdot (\mathbf{a} + \mathbf{b}) = k \cdot \mathbf{a} + k \cdot \mathbf{b}
  \]

- **P3** – The inner product is distributive with respect to sum:
  \[
  \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}
  \]

- **P4:**
  \[
  \mathbf{b} (\alpha + \beta) = \alpha \cdot \mathbf{b} + \beta \cdot \mathbf{b}
  \]
Exercise, P3:

- Prove that inner product is distributive with respect to sum:

\[ \vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} \]
Geometric Vectors: Reference Frames → Making Things Simpler

- Geometric vectors:
  - Easy to visualize but cumbersome to work with
  - The major drawback: hard to manipulate
    - Was very hard to carry out simple operations (recall proving the distributive property on previous slide)
    - When it comes to computers, which are good at storing matrices and vectors, having to deal with a geometric entity is cumbersome

- We are about to address these drawbacks by first introducing a Reference Frame (RF) in which we’ll express all our vectors
Basis (Unit Coordinate) Vectors

- Basis (Unit Coordinate) Vectors: a set of unit vectors used to express all other vectors of the 2D Euclidian space

- In this class, to simplify our life, we use a set of two orthonormal unit vectors
  - These two vectors, \( \mathbf{i} \) and \( \mathbf{j} \), define the x and y directions of the RF

- A vector \( \mathbf{a} \) can then be resolved into components \( a_x \) and \( a_y \), along the axes x and y:
  \[
  \mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j}
  \]

- Nomenclature: \( a_x \) and \( a_y \) are called the Cartesian components of the vector

- We’re going to exclusively work with right hand mutually orthogonal RFs
Recall the distributive property of the dot product

\[(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}\]

Based on the relation above, the following holds (expression for inner product when working in a reference frame):

\[
\vec{a} \cdot \vec{b} = (a_x \vec{i} + a_y \vec{j}) \cdot (b_x \vec{i} + b_y \vec{j}) = a_x b_x + a_y b_y
\]

Used to prove identity above (recall angle between basis vectors is \(\pi/2\)):

\[
\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = 1 \quad \vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{i} = 0
\]

Also, it’s easy to see that the projections \(a_x\) and \(a_y\) on the two axes are

\[
a_x = \vec{a} \cdot \vec{i} \quad a_y = \vec{a} \cdot \vec{j}
\]
Geometric Vectors: Loose Ends

- Given a vector \( \vec{a} \), the orthogonal vector \( \vec{a}^\perp \) is obtained as
  \[
  \vec{a} = a_x \vec{i} + a_y \vec{j} \quad \Rightarrow \quad \vec{a}^\perp = -a_y \vec{i} + a_x \vec{j} \quad \& \quad \vec{a} \cdot \vec{a}^\perp = 0
  \]

- Length of a vector expressed using Cartesian coordinates:
  \[
  \vec{a} \cdot \vec{a} = ||\vec{a}|| \cdot ||\vec{a}|| = a_x a_x + a_y a_y \quad \Rightarrow \quad ||\vec{a}|| = \sqrt{a_x^2 + a_y^2}
  \]

- Notation used:
  - Notation convention: throughout this class, vectors/matrices are in bold font, scalars are not (most often they are in italics)
New Concept: Algebraic Vectors

- Given a RF, each vector can be represented by a triplet

\[ \vec{a} = a_x \vec{i} + a_y \vec{j} \quad \Leftrightarrow \quad \vec{a} \mapsto (a_x, a_y) \]

- It doesn’t take too much imagination to associate to each geometric vector a two-dimensional algebraic vector:

\[ \vec{a} = a_x \vec{i} + a_y \vec{j} \quad \Leftrightarrow \quad a = \begin{bmatrix} a_x \\ a_y \end{bmatrix} \]

- Note that I dropped the arrow on \( \mathbf{a} \) to indicate that we are talking about an algebraic vector
Putting Things in Perspective...

- Step 1: We started with geometric vectors

- Step 2: We introduced a reference frame

- Step 3: Relative to that reference frame each geometric vector is uniquely represented as a pair of scalars (the Cartesian coordinates)

- Step 4: We generated an algebraic vector whose two entries are provided by the pair above
  - This vector is the algebraic representation of the geometric vector

- Note that the algebraic representations of the basis vectors are

\[
\begin{align*}
\vec{i} & \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \vec{j} & \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\end{align*}
\]
Fundamental Question:
How do G. Vector Ops. Translate into A. Vector Ops.?

- There is a straight correspondence between the operations
- Just a different representation of an old concept
  - Scaling a G. Vector $\Leftrightarrow$ Scaling of corresponding A. Vector
  - Adding two G. Vectors $\Leftrightarrow$ Adding the corresponding two A. Vectors
  - Inner product of two G. Vectors $\Leftrightarrow$ Dot Product of the two A. Vectors
    - We’ll talk about outer product later
  - Measure the angle $\theta$ between two G. Vectors $\rightarrow$ uses inner product, so
    it is based on the dot product of the corresponding A. Vectors
Algebraic Vector and Reference Frames

- Recall that an algebraic vector is just a representation of a geometric vector in a particular reference frame (RF)

- Question: What if I now want to represent the same geometric vector in a different RF?
Algebraic Vector and Reference Frames

- Representing the same geometric vector in a different RF leads to the concept of Rotation Matrix $A$:

- Getting the new coordinates, that is, representation of the same geometric vector in the new RF is as simple as multiplying the coordinates by the rotation matrix $A$:

$$s = As'$$

- NOTE 1: what is changed is the RF used for representing the vector, and not the underlying geometric vector

- NOTE 2: rotation matrix $A$ is sometimes called “orientation matrix”
The Rotation Matrix $A$

**Figure 2.4.1** Two Cartesian reference frames.

**Figure 2.4.2** Vector $\vec{s}$ in two frames.

$$A = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix}$$

- Very important observation $\rightarrow$ the matrix $A$ is orthonormal:

$$A^T A = A A^T = I_{2 \times 2}$$
Important Relation

- Expressing a given vector in one reference frame (local) in a different reference frame (global)

\[ s = As' \]

Also called a change of base.
Example 1

- Express the geometric vector $\overrightarrow{O'B}$ in the local reference frame $O’X’Y’$.
- Express the same geometric vector in the global reference frame $OXY$.
- Do the same for the geometric vector $\overrightarrow{O'E}$.
Example 2

- Express the geometric vector $\overrightarrow{O'G}$ in the local reference frame $O'X'Y'$.
- Express the same geometric vector in the global reference frame $OXY$.
- Do the same for the geometric vector $\overrightarrow{O'P}$. 