ME451
Kinematics and Dynamics of Machine Systems

Review of Linear Algebra

2.1 through 2.4

Tu, Sept. 07

Before we get started…

- Last time:
  - Syllabus
  - Quick overview of course
  - Starting discussion about vectors, their geometric representation

- HW Assigned:
  - ADAMS assignment, will be emailed to you today
  - Problems: 2.2.5, 2.2.8. 2.2.10
  - Due in one week

Geometric Entities: Their Relevance

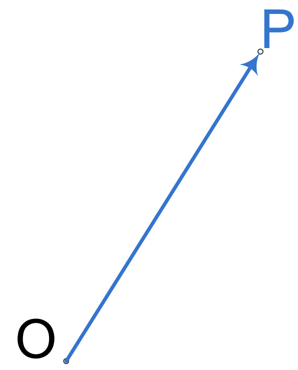
- Kinematics (and later Dynamics) of systems of rigid bodies:
  - Requires the ability to describe the position, velocity, and acceleration of each rigid body in the system as functions of time

  - In the Euclidian 2D space, geometric vectors and 2X2 orthonormal matrices are extensively used to this end

    - Geometric vectors - used to locate points on a body or the center of mass of a rigid body

    - 2X2 orthonormal matrices - used to describe the orientation of a body

Geometric Vectors



- What is a "Geometric Vector"?
  - A quantity that has three attributes:
    - A direction (given by the blue line)
    - A sense (from O to P)
    - A magnitude, ||OP||
  - Note that all geometric vectors are defined in relation to an origin **O**

- IMPORTANT:
  - Geometric vectors are entities that are independent of any reference frame

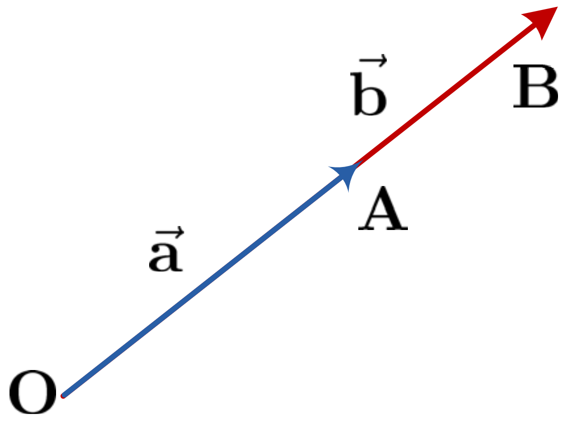
- ME451 deals <u>planar</u> kinematics and dynamics
  - We assume that all the vectors are defined in the 2D Euclidian space
  - A basis for the Euclidian space is any collection of two independent vectors

Geometric Vectors: Operations

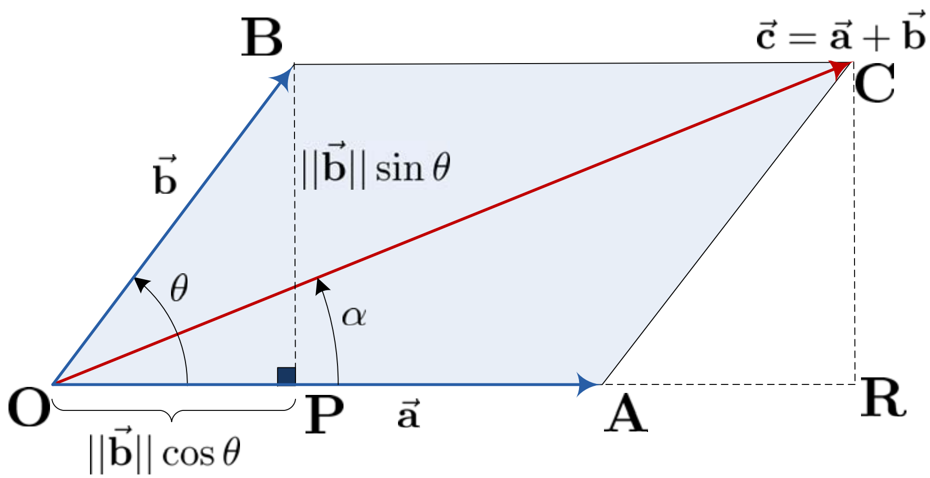
- What geometric vectors operations are <u>defined</u> out there?
  - Scaling by a scalar α
  - Addition of geometric vectors (the parallelogram rule)
  - Multiplication of two geometric vectors
    - The inner product rule (leads to a number)
    - The outer product rule (leads to a vector)
  - One can measure the angle θ between two geometric vectors
- A review these definitions follows over the next couple of slides

G. Vector Operation : Scaling by α

- By definition, scaling one geometric vector $\vec{\mathbf{a}}$ by a scalar $\alpha \neq 0$ leads to a new vector $\vec{\mathbf{b}} \equiv \alpha\vec{\mathbf{a}}$ that has the following three attributes:
  - $\vec{\mathbf{b}}$ has the same direction as the vector $\vec{\mathbf{a}}$
  - $\vec{\mathbf{b}}$ has the sense of $\vec{\mathbf{a}}$ if $\alpha > 0$, and opposite sense if $\alpha < 0$
  - The magnitude of $\vec{\mathbf{b}}$ is $b = |\alpha| a$
- Note that if $\alpha = 0$, then $\vec{\mathbf{b}}$ is the null vector.



G. Vector Operation: Addition of Two G. Vectors



- Sum of two vectors (definition)
  - Obtained by the parallelogram rule
- Operation is commutative
- Easy to visualize, pretty messy to summarize in an analytical fashion:

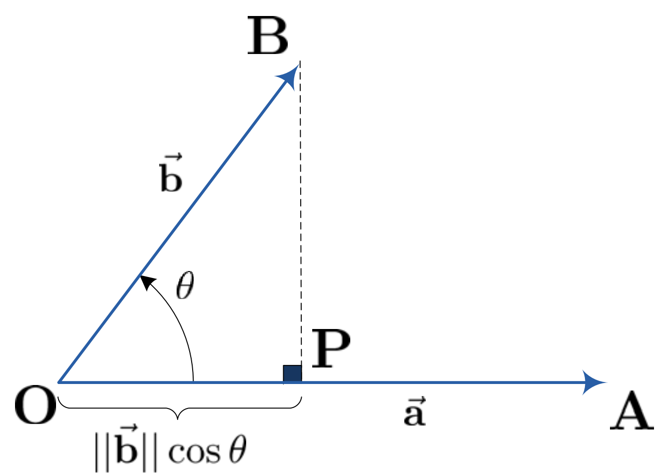
$$c = \sqrt{||\mathbf{OR}||^2 + ||\mathbf{RC}||^2} = \sqrt{(a + b\cos\theta)^2 + b^2\sin^2\theta}$$

$$\alpha = \tan^{-1}\frac{b\sin\theta}{a + b\cos\theta}$$

G. Vector Operation: Inner Product of Two G. Vectors

- The product between the magnitude of the first geometric vector and the projection of the second vector onto the first vector

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = ||\vec{\mathbf{a}}|| \cdot ||\vec{\mathbf{b}}|| \cdot \cos(\vec{\mathbf{a}}, \vec{\mathbf{b}})$$



- Note that operation is commutative

- Don't call this the "dot product" of the two vectors
  - This name is saved for algebraic vectors

G. Vector Operation: Angle Between Two G. Vectors

- Regarding the angle between two vectors, note that

$$\theta(\vec{\mathbf{a}}, \vec{\mathbf{b}}) \neq \theta(\vec{\mathbf{b}}, \vec{\mathbf{a}}) \qquad\qquad \theta(\vec{\mathbf{a}}, \vec{\mathbf{b}}) = 2\pi - \theta(\vec{\mathbf{b}}, \vec{\mathbf{a}})$$

- Important: Angles are positive <u>counterclockwise</u>
  - This is why when measuring the angle between two vectors it's important what the first (start) vector is

Combining Basic G. Vector Operations

- P1 – The sum of geometric vectors is associative

$$\vec{\mathbf{a}} + (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = (\vec{\mathbf{a}} + \vec{\mathbf{b}}) + \vec{\mathbf{c}}$$

- P2 – Multiplication with a scalar is distributive with respect to the sum:

$$k \cdot (\vec{\mathbf{a}} + \vec{\mathbf{b}}) = k \cdot \vec{\mathbf{a}} + k \cdot \vec{\mathbf{b}}$$

- P3 – The inner product is distributive with respect to sum:

$$\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} + \vec{\mathbf{a}} \cdot \vec{\mathbf{c}}$$

- P4:

$$\vec{\mathbf{b}}(\alpha + \beta) = \alpha \cdot \vec{\mathbf{b}} + \beta \cdot \vec{\mathbf{b}}$$

[AO]

Exercise, P3:

- Prove that inner product is distributive with respect to sum:

$$\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}+\vec{\mathbf{c}})=\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}+\vec{\mathbf{a}}\cdot\vec{\mathbf{c}}$$

Geometric Vectors: Reference Frames → Making Things Simpler

- Geometric vectors:
  - Easy to visualize but cumbersome to work with
  - The major drawback: hard to manipulate
    - Was very hard to carry out simple operations (recall proving the distributive property on previous slide)
    - When it comes to computers, which are good at storing matrices and vectors, having to deal with a geometric entity is cumbersome

- We are about to address these drawbacks by first introducing a Reference Frame (RF) in which we'll express all our vectors

Basis (Unit Coordinate) Vectors

- Basis (Unit Coordinate) Vectors: a set of unit vectors used to express all other vectors of the 2D Euclidian space

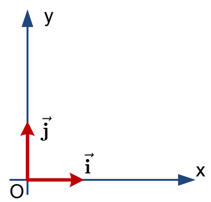
- In this class, to simplify our life, we use a set of two orthonormal unit vectors
  - These two vectors, $\vec{\mathbf{i}}$ and $\vec{\mathbf{j}}$, define the x and y directions of the RF

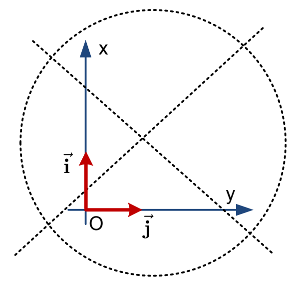
- A vector **a** can then be resolved into components a_x and a_y , along the axes *x* and *y* :

$$\vec{\mathbf{a}} = a_x\vec{\mathbf{i}} + a_y\vec{\mathbf{j}}$$

- Nomenclature: a_x and a_y are called the Cartesian components of the vector

- We're going to exclusively work with right hand mutually orthogonal RFs





Geometric Vectors: Operations

- Recall the distributive property of the dot product

$$\left(\vec{\mathbf{a}} + \vec{\mathbf{b}}\right) \cdot \vec{\mathbf{c}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{c}} + \vec{\mathbf{b}} \cdot \vec{\mathbf{c}}$$

- Based on the relation above, the following holds (expression for inner product when working in a reference frame):

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \left(a_x \vec{i} + a_y \vec{j}\right) \cdot \left(b_x \vec{i} + b_y \vec{j}\right) = a_x\, b_x + a_y\, b_y$$

- Used to prove identity above (recall angle between basis vectors is $\pi/2$):

$$\vec{\mathbf{i}} \cdot \vec{\mathbf{i}} = \vec{\mathbf{j}} \cdot \vec{\mathbf{j}} = 1 \qquad \vec{\mathbf{i}} \cdot \vec{\mathbf{j}} = \vec{\mathbf{j}} \cdot \vec{\mathbf{i}} = 0$$

- Also, it's easy to see that the projections ax and ay on the two axes are

$$a_x = \vec{\mathbf{a}} \cdot \vec{\mathbf{i}} \qquad a_y = \vec{\mathbf{a}} \cdot \vec{\mathbf{j}}$$

Geometric Vectors: Loose Ends

- Given a vector $\vec{\mathbf{a}}$, the orthogonal vector $\vec{\mathbf{a}}^{\perp}$ is obtained as

$$\vec{\mathbf{a}} = a_x\vec{i} + a_y\vec{j} \qquad \Rightarrow \qquad \vec{\mathbf{a}}^{\perp} = -a_y\vec{i} + a_x\vec{j} \qquad \& \qquad \vec{\mathbf{a}} \cdot \vec{\mathbf{a}}^{\perp} = 0$$

- Length of a vector expressed using Cartesian coordinates:

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{a}} = ||\vec{\mathbf{a}}|| \cdot ||\vec{\mathbf{a}}|| = a_x a_x + a_y a_y \qquad \Rightarrow \qquad ||\vec{\mathbf{a}}|| = \sqrt{a_x^2 + a_y^2}$$

- Notation used:
  - Notation convention: throughout this class, vectors/matrices are in bold font, scalars are not (most often they are in italics)

New Concept: Algebraic Vectors

- Given a RF, each vector can be represented by a triplet

$$\vec{\mathbf{a}} = a_x \vec{\mathbf{i}} + a_y \vec{\mathbf{j}} \qquad \Leftrightarrow \qquad \vec{\mathbf{a}} \mapsto (a_x, a_y)$$

- It doesn't take too much imagination to associate to each <u>geometric</u> vector a two-dimensional <u>algebraic</u> vector:

$$\vec{\mathbf{a}} = a_x \vec{\mathbf{i}} + a_y \vec{\mathbf{j}} \qquad \Leftrightarrow \qquad \mathbf{a} = \begin{bmatrix} a_x \\ a_y \end{bmatrix}$$

- Note that I dropped the arrow on **a** to indicate that we are talking about an algebraic vector

Putting Things in Perspective…

- Step 1: We started with geometric vectors
- Step 2: We introduced a reference frame
- Step 3: Relative to that reference frame each geometric vector is uniquely represented as a pair of scalars (the Cartesian coordinates)
- Step 4: We generated an algebraic vector whose two entries are provided by the pair above
  - This vector is the algebraic representation of the geometric vector
- Note that the algebraic representations of the basis vectors are

$$\vec{\mathbf{i}} \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \vec{\mathbf{j}} \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Fundamental Question:
How do G. Vector Ops. Translate into A. Vector Ops.?

- There is a straight correspondence between the operations
- Just a different representation of an old concept
  - Scaling a G. Vector ⇔ Scaling of corresponding A. Vector
  - Adding two G. Vectors ⇔ Adding the corresponding two A. Vectors
  - Inner product of two G. Vectors ⇔ Dot Product of the two A. Vectors
    - We'll talk about outer product later
  - Measure the angle θ between two G. Vectors $\rightarrow$ uses inner product, so it is based on the dot product of the corresponding A. Vectors

Algebraic Vector and Reference Frames

- Recall that an algebraic vector is just a representation of a geometric vector in a particular reference frame (RF)

- Question: What if I now want to represent the same geometric vector in a different RF?

Algebraic Vector and Reference Frames

- Representing the same geometric vector in a different RF leads to the concept of Rotation Matrix **A**:
  - Getting the new coordinates, that is, representation of the **same** geometric vector in the new RF is as simple as multiplying the coordinates by the rotation matrix **A**:

$$\mathbf{s} = \mathbf{A}\mathbf{s}'$$

- NOTE 1: what is changed is the RF used for representing the vector, and not the underlying geometric vector
- NOTE 2: rotation matrix **A** is sometimes called "orientation matrix"

The Rotation Matrix A

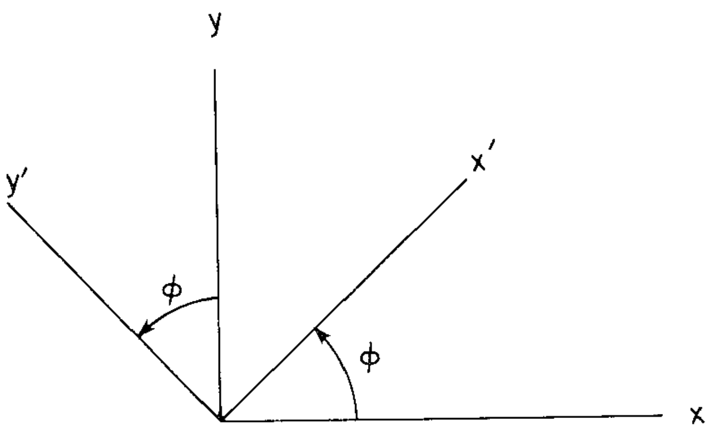


Figure 2.4.1 Two Cartesian reference frames.

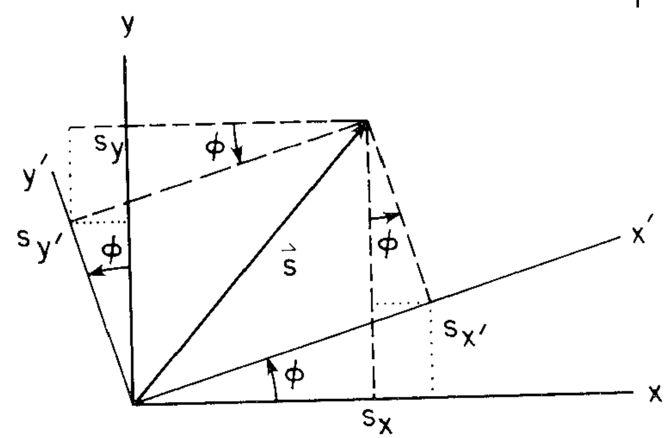


Figure 2.4.2 Vector $\vec{s}$ in two frames.

$$\mathbf{A} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

- Very important observation $\rightarrow$ the matrix **A** is orthonormal:

$$\mathbf{A}^T\,\mathbf{A} = \mathbf{A}\,\mathbf{A}^T = \mathbf{I}_{2\times 2}$$

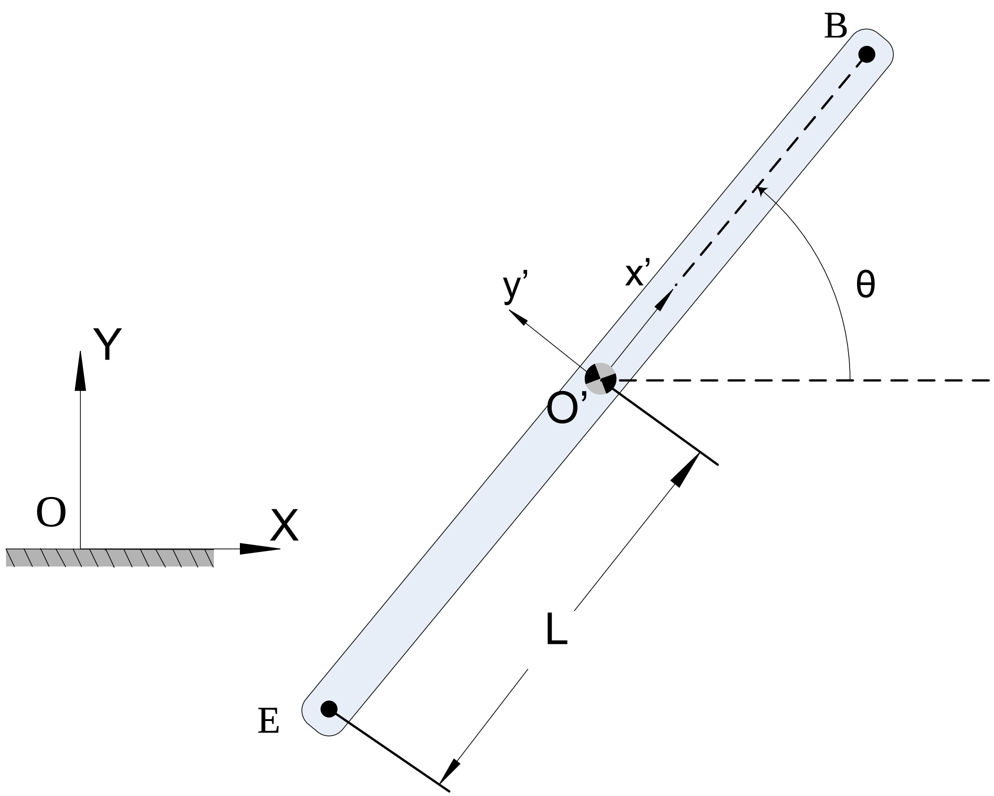
Important Relation

- Expressing a given vector in one reference frame (local) in a different reference frame (global)

$$\mathbf{s} = \mathbf{A}\mathbf{s}'$$

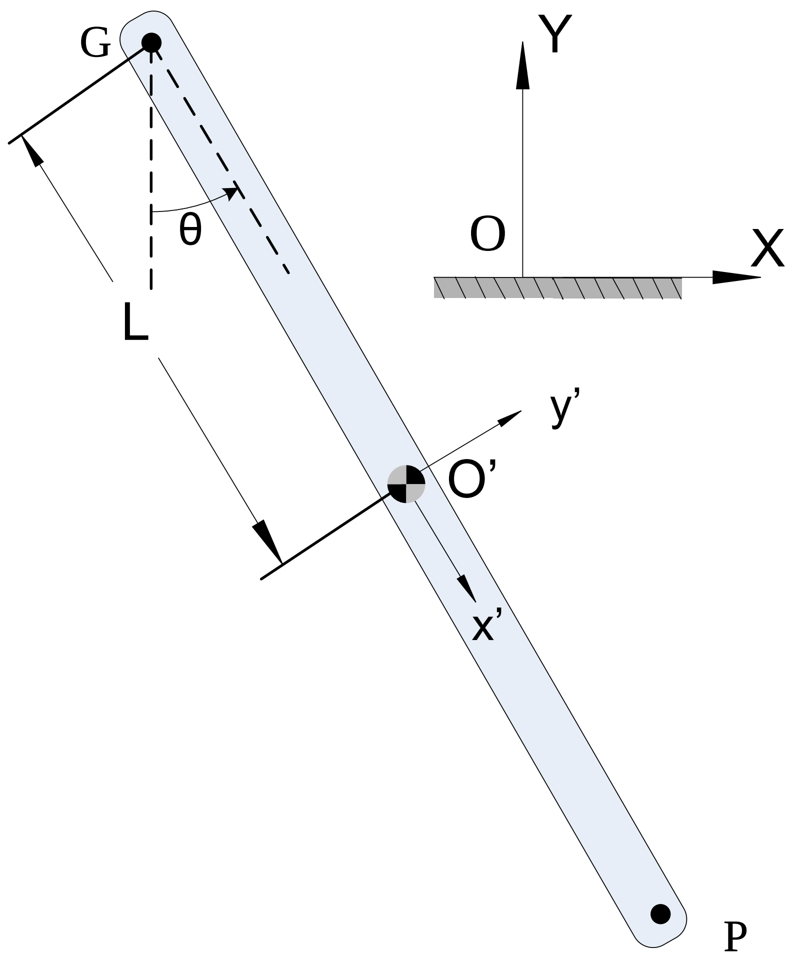
Also called a change of base.

Example 1



- Express the geometric vector $\overrightarrow{O'B}$ in the local reference frame O'X'Y'.
- Express the same geometric vector in the global reference frame OXY
- Do the same for the geometric vector $\overrightarrow{O'E}$

Example 2



- Express the geometric vector

  $$\overrightarrow{O'P}$$

  in the local reference frame O'X'Y'.
- Express the same geometric vector in the global reference frame OXY
- Do the same for the geometric vector

  $$\overrightarrow{O'G}$$