

# Example: Equation of Motion in slider Crank Mechanism

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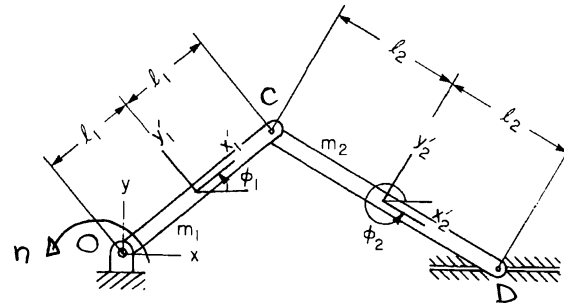


Figure 6.3.2 Two-body slider-crank.

The EOM for any mechanism assume the form:

$$M \ddot{q} + \phi_q^T \lambda = Q^A$$

For our mechanism, we need to identify the expression

of  $M$ ,  $\phi$ ,  $\phi_q$ ,  $Q^A$ .

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ \phi_1 \\ r_2 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ \phi_1 \\ x_2 \\ y_2 \\ \phi_2 \end{bmatrix}$$

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} m_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & J_1' & 0 & 0 & 0 \\ 0 & 0 & 0 & m_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & J_2' \end{bmatrix}$$

As far as the constraints are concerned, we have

- Revolute joint between Body 1 and ground
- Revolute joint between Body 1 and Body 2.
- y absolute constraint for point D on body 2

$$r_1^D = r_1 + A_1 s_1^D = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \end{bmatrix} \begin{bmatrix} -l_1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 - l_1 c\phi_1 \\ y_1 - l_1 s\phi_1 \end{bmatrix}$$

$$r_1^C = r_1 + A_1 s_1^C = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \end{bmatrix} \begin{bmatrix} l_1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 + l_1 c\phi_1 \\ y_1 + l_1 s\phi_1 \end{bmatrix}$$

$$r_2^C = r_2 + A_2 s_2^C = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} c\phi_2 & -s\phi_2 \\ s\phi_2 & c\phi_2 \end{bmatrix} \begin{bmatrix} -l_2 \\ 0 \end{bmatrix} = \begin{bmatrix} x_2 - l_2 c\phi_2 \\ y_2 - l_2 s\phi_2 \end{bmatrix}$$

$$r_2^D = r_2 + A_2 s_2^D = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} c\phi_2 & -s\phi_2 \\ s\phi_2 & c\phi_2 \end{bmatrix} \begin{bmatrix} l_2 \\ 0 \end{bmatrix} = \begin{bmatrix} x_2 + l_2 c\phi_2 \\ y_2 + l_2 s\phi_2 \end{bmatrix}$$

Then,

$$\phi(\gamma, t) = \begin{bmatrix} x_1 - l_1 c\phi_1 \\ y_1 - l_1 s\phi_1 \\ x_1 + l_1 c\phi_1 - x_2 + l_2 c\phi_2 \\ y_1 + l_1 s\phi_1 - y_2 + l_2 s\phi_2 \\ y_2 + l_2 s\phi_2 \end{bmatrix} = 0$$

$$\phi_q = \begin{bmatrix} 1 & 0 & l_1 s\phi_1 & 0 & 0 & 0 \\ 0 & 1 & -l_1 c\phi_1 & 0 & 0 & 0 \\ 1 & 0 & -l_1 s\phi_1 & -1 & 0 & -l_2 s\phi_2 \\ 0 & 1 & l_1 c\phi_1 & 0 & -1 & l_2 c\phi_2 \\ 0 & 0 & 0 & 0 & 1 & l_2 c\phi_2 \end{bmatrix}$$

Applied forces:

$$Q_1^A = \begin{bmatrix} 0 \\ -m_1 g \\ n \end{bmatrix}$$

$$Q_2^A = \begin{bmatrix} 0 \\ -m_2 g \\ 0 \end{bmatrix}$$

Note:  $n$  is a given applied torque.

Since we have 5 constraints, we'll have 5 Lagrange multipliers:

$$\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{bmatrix}$$

The equations of motion then assume the expression:

$$\begin{bmatrix} m_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & J_1' & 0 & 0 & 0 \\ 0 & 0 & 0 & m_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & J_2' \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\phi}_1 \\ \ddot{x}_2 \\ \ddot{y}_2 \\ \ddot{\phi}_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ l_1 s \phi_1 & -l_1 c \phi_1 & -l_1 s \phi_1 & l_1 c \phi_1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -l_2 s \phi_2 & l_2 c \phi_2 & l_2 c \phi_2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{bmatrix} = \begin{bmatrix} 0 \\ -m_1 g \\ n \\ 0 \\ -m_2 g \\ 0 \end{bmatrix}$$

This needs to be coupled with the acceleration equation to lead to a linear system of equations in the unknowns  $\ddot{q}$  and  $\lambda$ . The acceleration equation is


$$\phi_p \cdot \ddot{p} = \gamma$$

For our problem  $\gamma$  is computed below.

$$\dot{\phi} = \begin{bmatrix} \dot{x}_1 + l_1 \dot{\phi}_1 s\phi_1 \\ \dot{y}_1 - l_1 \dot{\phi}_1 c\phi_1 \\ \dot{x}_1 - l_1 \dot{\phi}_1 s\phi_1 - \dot{x}_2 - l_2 \dot{\phi}_2 s\phi_2 \\ \dot{y}_1 + l_1 \dot{\phi}_1 c\phi_1 - \dot{y}_2 + l_2 \dot{\phi}_2 c\phi_2 \\ \dot{y}_2 + l_2 \dot{\phi}_2 c\phi_2 \end{bmatrix} = 0$$

$$\ddot{\phi} = \begin{bmatrix} \ddot{x}_1 + \ddot{\phi}_1 l_1 s\phi_1 + l_1 \dot{\phi}_1^2 c\phi_1 \\ \ddot{y}_1 - \ddot{\phi}_1 l_1 c\phi_1 + l_1 \dot{\phi}_1^2 s\phi_1 \\ \ddot{x}_1 - \ddot{\phi}_1 l_1 s\phi_1 - l_1 \dot{\phi}_1^2 c\phi_1 - \ddot{x}_2 - l_2 \ddot{\phi}_2 s\phi_2 - l_2 \dot{\phi}_2^2 c\phi_2 \\ \ddot{y}_1 + \ddot{\phi}_1 l_1 c\phi_1 - l_1 \dot{\phi}_1^2 s\phi_1 - \ddot{y}_2 + \ddot{\phi}_2 l_2 c\phi_2 - l_2 \dot{\phi}_2^2 s\phi_2 \\ \ddot{y}_2 + \ddot{\phi}_2 l_2 c\phi_2 - l_2 \dot{\phi}_2^2 s\phi_2 \end{bmatrix} = 0$$

$$\Rightarrow \phi_p \cdot \ddot{q} = \begin{bmatrix} -l_1 \dot{\phi}_1^2 c\phi_1 \\ -l_1 \dot{\phi}_1^2 s\phi_1 \\ l_1 \dot{\phi}_1^2 c\phi_1 + l_2 \dot{\phi}_2^2 c\phi_2 \\ l_1 \dot{\phi}_1^2 s\phi_1 + l_2 \dot{\phi}_2^2 s\phi_2 \\ l_2 \dot{\phi}_2^2 s\phi_2 \end{bmatrix}$$


  
 $g$

The constrained eom will then assume the form:

