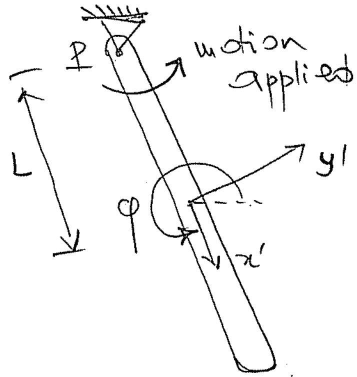
Example

~ simple pendulum ~



A torque is applied to drive the pendulum like $\varphi = \frac{3\pi}{2} + 2\pi t$.

Mass $\rightarrow m$

Mass moment of inertia $\rightarrow J'$

$$M = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J' \end{bmatrix}$$

Constraints (kinematic and driving):

because rev. joint is at the origin.

$$r^P = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} C\varphi & -S\varphi \\ S\varphi & C\varphi \end{bmatrix} \begin{bmatrix} -L \\ 0 \end{bmatrix} = \begin{bmatrix} x - LC\varphi \\ y + LS\varphi \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then,

$$\Phi(q,t) = \begin{bmatrix} x - LC\varphi \\ y - LS\varphi \\ \varphi - 2\pi t - \frac{3\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \Phi_q = \begin{bmatrix} 1 & 0 & LS\varphi \\ 0 & 1 & -LC\varphi \\ 0 & 0 & 1 \end{bmatrix}$$

Generalized applied force: $Q^A = \begin{bmatrix} 0 \\ -mg \\ 0 \end{bmatrix}$

Then, the EOM assumes the form (based on $M\ddot{q} + \Phi_q^T \lambda = Q^A$)

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J' \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y}' \\ \ddot{\varphi} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ LS\varphi & -LC\varphi & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -mg \\ 0 \end{bmatrix}$$

I need the acceleration constraint equation:

$$\dot{\Phi}(q,t) = \begin{bmatrix} \dot{x} + \dot{\varphi} LS\varphi \\ \dot{y} - \dot{\varphi} LC\varphi \\ \dot{\varphi} - 2\pi \end{bmatrix} \qquad \ddot{\Phi}(q,t) = \begin{bmatrix} \ddot{x} + \ddot{\varphi} LS\varphi + \dot{\varphi}^2 LC\varphi \\ \ddot{y} - \ddot{\varphi} LC\varphi + \dot{\varphi}^2 LS\varphi \\ \ddot{\varphi} \end{bmatrix}$$

Therefore, the RHS of the acceleration equation is

$$\gamma = \begin{bmatrix} -\dot{\varphi}^2 L c\varphi \\ -\dot{\varphi}^2 L s\varphi \\ 0 \end{bmatrix}$$

We can now assemble the linear system that helps us get the acceleration and λ:

$$\begin{array}{c} \qquad M \qquad\qquad\qquad \Phi_q^T \\ \left[\begin{array}{ccc|ccc} m & 0 & 0 & 1 & 0 & 0 \\ 0 & m & 0 & 0 & 1 & 0 \\ 0 & 0 & J' & Ls\varphi & -Lc\varphi & 1 \\ \hline 1 & 0 & Ls\varphi & 0 & 0 & 0 \\ 0 & 1 & -Lc\varphi & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \\ \Phi_q \end{array} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\varphi} \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -mg \\ 0 \\ -\dot{\varphi}^2 Lc\varphi \\ -\dot{\varphi}^2 Ls\varphi \\ 0 \end{bmatrix}$$

Fortunately enough, you can solve this linear system without resorting to a numerical approach. You get:

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\varphi}_1 \end{bmatrix} = \begin{bmatrix} -L\cos\varphi \cdot \dot{\varphi}^2 \\ -Ls\varphi \cdot \dot{\varphi}^2 \\ 0 \end{bmatrix} \quad \oplus \quad \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} mL\dot{\varphi}^2 \cdot c\varphi \\ -mg + mL\dot{\varphi}^2 s\varphi \\ -mgL \cdot c\varphi \end{bmatrix}$$

Next, let's compute the reaction forces induced by this pin joint.

To this end, let's identify first Φ_r and Φ_φ:

$$\Phi_r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \Phi_\varphi = \begin{bmatrix} Ls\varphi \\ -Lc\varphi \\ 1 \end{bmatrix}$$

According to what we discussed in class,

$$F^P = -\Phi_r^T \lambda \qquad \& \qquad T = -s'^{PT} \cdot B^T \cdot F^P - \Phi_\varphi^T \cdot \lambda$$

Then,

$$F^{P} = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} mL\dot{\varphi}^2 c\varphi \\ -mg + mL\dot{\varphi}^2 s\varphi \\ -mgLc\varphi \end{bmatrix} = -\begin{bmatrix} mL\dot{\varphi}^2 c\varphi \\ -mg + mL\dot{\varphi}^2 s\varphi \end{bmatrix}$$

For the reaction torque:

$$T = [-L \quad 0]\cdot\begin{bmatrix} -s\varphi & c\varphi \\ -c\varphi & -s\varphi \end{bmatrix}\cdot\begin{bmatrix} mL\dot{\varphi}^2 c\varphi \\ -mg + mL\dot{\varphi}^2 s\varphi \end{bmatrix} - [Ls\varphi \;\; -Lc\varphi \;\; 1]\begin{bmatrix} mL\dot{\varphi}^2 c\varphi \\ -mg + mL\dot{\varphi}^2 s\varphi \\ -mgLc\varphi \end{bmatrix}$$

$$= [Ls\varphi \quad -Lc\varphi]\begin{bmatrix} mL\dot{\varphi}^2 c\varphi \\ -mg + mL\dot{\varphi}^2 s\varphi \end{bmatrix} - \cancel{mL^2\dot{\varphi}^2 s\varphi c\varphi} - mgLc\varphi$$

$$+ \cancel{mL^2\dot{\varphi}^2 s\varphi c\varphi} + mgLc\varphi = mgLc\varphi$$

$$\Rightarrow \quad F^{P} = -\begin{bmatrix} mL\dot{\varphi}^2 c\varphi \\ -mg + mL\dot{\varphi}^2 s\varphi \end{bmatrix} \qquad T = mgLc\varphi.$$

Quick Remark:

For this simple problem, you know that $\varphi = 2\pi t + \frac{3\pi}{2}$

$\Rightarrow \dot{\varphi} = 2\pi.$

Then:

$$F^{P}(t) = \begin{bmatrix} -mL(4\pi)^2\cdot\cos(2\pi t + \frac{3\pi}{2}) \\ mg - mL(4\pi)^2\sin(2\pi t + \frac{3\pi}{2}) \end{bmatrix} \quad \& \quad T = mgL\cos(2\pi t + \tfrac{3\pi}{2})$$