



$m = 30\text{kg}$
 $J = 2.5\text{ kg}\cdot\text{m}^2$
 $k = 8\frac{\text{N}\cdot\text{m}}{\text{rad}}$
 $c = 1\frac{\text{N}\cdot\text{m}\cdot\text{s}}{\text{rad}}$

spring zero tension angle $\phi_{\text{free}} = 0$.

Generalized coordinates used:

$$q = \begin{bmatrix} x \\ y \\ \phi \end{bmatrix}$$

constraints:

A) kinematic: revolute joint @ hinge (origin of global reference frame)

$$r^0 = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c\phi & -s\phi \\ s\phi & c\phi \end{bmatrix} \begin{bmatrix} -L \\ 0 \end{bmatrix} = \begin{bmatrix} x - Lc\phi \\ y - Ls\phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

B) driving constraint on angle ϕ :

$$\Phi^D(q, t) = \phi - \frac{\pi}{2} \sin\left(\frac{\pi}{4}t\right) = 0$$

you want this motion for the first two seconds.

Therefore:

$$\Phi(q, t) = \begin{bmatrix} x - Lc\phi \\ y - Ls\phi \\ \phi - \frac{\pi}{2} \sin\left(\frac{\pi}{4}t\right) \end{bmatrix} \Rightarrow \Phi_q = \begin{bmatrix} 1 & 0 & Ls\phi \\ 0 & 1 & -Lc\phi \\ 0 & 0 & 1 \end{bmatrix}$$

$$\dot{\Phi}(q, t) = \begin{bmatrix} \dot{x} + \dot{\phi} Ls\phi \\ \dot{y} - \dot{\phi} Lc\phi \\ \dot{\phi} - \frac{\pi^2}{8} \cos\left(\frac{\pi}{4}t\right) \end{bmatrix} \Rightarrow \dot{\Phi} = \begin{bmatrix} 0 \\ 0 \\ \frac{\pi^2}{8} \cos\left(\frac{\pi}{4}t\right) \end{bmatrix}$$

$$\ddot{\Phi}(p, t) = \begin{bmatrix} \ddot{x} + \dot{\varphi}^2 L_S \varphi + \ddot{\varphi}^2 L_C \varphi \\ \ddot{y} - \dot{\varphi}^2 L_C \varphi + \ddot{\varphi}^2 L_S \varphi \\ \ddot{\varphi} + \frac{u^3}{32} \sin\left(\frac{z}{4} t\right) \end{bmatrix} \Rightarrow \gamma = \begin{bmatrix} -\dot{\varphi}^2 L_C \varphi \\ -\dot{\varphi}^2 L_S \varphi \\ -\frac{u^3}{32} \sin\left(\frac{z}{4} t\right) \end{bmatrix}$$

Generalized forces:

- Gravity is irrelevant in this case. The only non-zero term in the expression of the generalized force is coming out of the rotational spring-damper element present in the model:

$$n = -k(\varphi - \varphi_{\text{free}}) - c\dot{\varphi}$$

$$\Rightarrow n = -(k\varphi + c\dot{\varphi})$$

Then, we conclude that $Q^A = \begin{bmatrix} 0 \\ 0 \\ -k\varphi - c\dot{\varphi} \end{bmatrix}$

We have all the quantities needed to solve the inverse dynamics problem:

$$\ddot{\Phi}_q \cdot \ddot{q} = \gamma \Rightarrow \begin{bmatrix} 1 & 0 & L_S \varphi \\ 0 & 1 & -L_C \varphi \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\varphi} \end{bmatrix} = \begin{bmatrix} -\dot{\varphi}^2 L_C \varphi \\ -\dot{\varphi}^2 L_S \varphi \\ -\frac{u^3}{32} \sin\left(\frac{z}{4} t\right) \end{bmatrix} \quad (*)$$

As far as the time evolution of φ , based on the expression of φ and the last equation in $\ddot{\Phi}_q \cdot \ddot{q} = \gamma$ and $\ddot{\Phi}_q \cdot \ddot{q} = \gamma$, we get

$$\begin{aligned} \varphi &= \frac{u}{2} \sin\left(\frac{z}{4} t\right) \\ \dot{\varphi} &= \frac{u^2}{8} \cos\left(\frac{z}{4} t\right) \\ \ddot{\varphi} &= -\frac{u^3}{32} \sin\left(\frac{z}{4} t\right). \end{aligned} \quad (**)$$

Next, based on Eq. (*) (see previous page),

$$\ddot{x} + L \dot{\varphi} s\varphi = -\dot{\varphi}^2 L r\varphi \quad \Rightarrow \quad \ddot{x} = -L \dot{\varphi} s\varphi - \dot{\varphi}^2 L r\varphi$$

$$\ddot{\varphi} - L \dot{\varphi} c\varphi = -\dot{\varphi}^2 L s\varphi \quad \Rightarrow \quad \ddot{\varphi} = L \dot{\varphi} c\varphi - \dot{\varphi}^2 L s\varphi.$$

Therefore, based on (**), we obtain the expression of \ddot{x} and $\ddot{\varphi}$ as functions of time. We therefore know how \dot{q} changes in time and are ready to compute the Lagrange multipliers. Before we do that, first note that we also know q and \dot{q} :

$$q = \begin{bmatrix} x \\ y \\ \varphi \end{bmatrix} = \begin{bmatrix} L r\varphi \\ L s\varphi \\ \frac{\pi}{2} \cdot \sin\left(\frac{\pi}{4} t\right) \end{bmatrix}$$

$$\dot{q} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} -\dot{\varphi} L s\varphi \\ \dot{\varphi} L r\varphi \\ \frac{\pi}{8} \cos\left(\frac{\pi}{4} t\right) \end{bmatrix}$$

Next, write the EOM:

$$M \ddot{q} + \Phi_g^T \lambda = Q^A$$

$$M = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J' \end{bmatrix}$$

$$Q^A = \begin{bmatrix} 0 \\ 0 \\ -k\varphi - c\dot{\varphi} \end{bmatrix}$$

Then, $\Phi_g^T \lambda = Q^A - M \ddot{q} = \begin{bmatrix} 0 \\ 0 \\ -k\varphi - c\dot{\varphi} \end{bmatrix} - \begin{bmatrix} m \ddot{x} \\ m \ddot{y} \\ J' \ddot{\varphi} \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ L s\varphi & -L r\varphi & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} -m \ddot{x} \\ -m \ddot{y} \\ -k\varphi - c\dot{\varphi} - J' \ddot{\varphi} \end{bmatrix}$$

Then,

$$\lambda_1 = -m \ddot{x} = m (L \ddot{\varphi} \sin \varphi + \dot{\varphi}^2 L \cos \varphi)$$

$$\lambda_2 = -m \ddot{y} = m (-L \ddot{\varphi} \cos \varphi + \dot{\varphi}^2 L \sin \varphi)$$

$$\begin{aligned} \lambda_3 &= -k\varphi - c\dot{\varphi} - J' \ddot{\varphi} - L \sin \varphi \cdot \lambda_1 + L \cos \varphi \lambda_2 \\ &= -k\varphi - c \cdot \dot{\varphi} - J' \ddot{\varphi} - L \sin \varphi m (L \ddot{\varphi} \sin \varphi + \dot{\varphi}^2 L \cos \varphi) + L \cos \varphi m (-L \ddot{\varphi} \cos \varphi + \dot{\varphi}^2 L \sin \varphi) \\ &= -k\varphi - c\dot{\varphi} - J' \ddot{\varphi} - mL^2 \ddot{\varphi} = -[(mL^2 + J') \ddot{\varphi} + c\dot{\varphi} + k\varphi] \end{aligned}$$

For our problem $c = 1$ & $k = 8$

We are interested in the reaction force/torque associated with the third constraint (the "driving" constraint).

$$\Phi^D(\varphi, t) = \varphi - \frac{\pi}{2} \cdot \sin\left(\frac{\pi}{4}t\right)$$

$$\Rightarrow \Phi_r^D = [0 \quad 0] \quad \& \quad \Phi_\varphi^D = [1].$$

$$\text{Then } F^D = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \& \quad T^D = -(\Phi_\varphi^D)^T \lambda_3 = -\lambda_3$$

$$\Rightarrow T^D = (mL^2 + J') \ddot{\varphi} + c\dot{\varphi} + k\varphi$$

$$= [30 \cdot (0.25) + 2.5] \cdot \left(-\frac{\pi^3}{32} \sin\left(\frac{\pi}{4}t\right)\right) + \frac{\pi^2}{8} \cos\left(\frac{\pi}{4}t\right) + 4\pi \cdot \sin\left(\frac{\pi}{4}t\right)$$

$$= \left(4\pi - \frac{10\pi^3}{32}\right) \sin\left(\frac{\pi}{4}t\right) + \frac{\pi^2}{8} \cos\left(\frac{\pi}{4}t\right).$$

$$\approx 2.877 \sin\left(\frac{\pi}{4}t\right) + 1.234 \cos\left(\frac{\pi}{4}t\right) \quad [\text{N} \cdot \text{m}].$$

Apply this torque for the first two seconds to get the specified time evolution on $\varphi = \frac{\pi}{2} \cdot \sin\left(\frac{\pi}{4}t\right)$.

