

Figure 3.5.10 Wrecker boom with a translational-distance driver.

Generalized coordinates:

$$q_1 = \begin{bmatrix} x_1 \\ y_1 \\ \phi_1 \end{bmatrix}$$

$$q_2 = \begin{bmatrix} x_2 \\ y_2 \\ \phi_2 \end{bmatrix}$$

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ \phi_1 \\ r_2 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ \phi_1 \\ x_2 \\ y_2 \\ \phi_2 \end{bmatrix}$$

STEP A: Identify joints and drivers

1 Absolute position constraint of point P1 on body 1 (x & y components)

$$c_1 = 0 \quad c_2 = 0 \quad \Sigma_1^P = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (\text{see Eq. 3.2.3 \& 3.2.4})$$

2 Rotational driver between Body 1 and Ground. This is an absolute driver constraint, use Eq 3.5.3.

$$c_3(t) = 0.025t$$

3 Translational joint between Body 1 & 2.

v1' - defined by P1 and Q1 : s1^P = [0; 0] s1^Q = [1; 0]

v2' - defined by P2 and Q2 : s2^P = [0; 0] s2^Q = [1; 0]

v1' = s1^Q - s1^P = [1; 0] v2' = s2^Q - s2^P = [1; 0]

(SEE Eq. 3.3.13)

4 Translational distance driver, between Body 1 & 2.

In our case ||v1'|| = ||v2'|| = 1 (the vectors have norm 1).

v1' & v2' are defined like in 3 above.

(SEE Eq. 3.5.12).

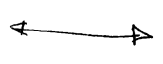
STEP B

Getting together phi(q, t).

phi^1 -> use Eqs. 3.2.3 & 3.2.4.

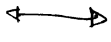
phi^1(q) = R1^P - R0^P = [x1; y1] + A1 [0; 0] - [0; 0] = [0; 0]

Therefore, phi^1(q) = [x1; y1].



phi^2 -> use Eq 3.5.3

$$\begin{aligned} \phi^{(2)}(q, t) &= \phi_1 - c_3(t) \\ &= \phi_1 - 0.025t = 0. \end{aligned}$$



$$\phi^{(3)}(q) \rightarrow \text{use Eq. 3.3.13}$$

$$\phi^{(3)}(q) = \begin{bmatrix} v_1'^T B_{11}^T (r_2 - r_1) - v_1'^T B_{12} s_2'^P - v_1'^T R^T s_1'^P \\ - v_1'^T B_{12} v_2' \end{bmatrix}$$

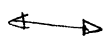
$$B_{11} v_1' = \begin{bmatrix} -s\phi_1 & -c\phi_1 \\ c\phi_1 & -s\phi_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -s\phi_1 \\ c\phi_1 \end{bmatrix}$$

$$B_{12} v_2' = \begin{bmatrix} -s(\phi_2 - \phi_1) & -c(\phi_2 - \phi_1) \\ c(\phi_2 - \phi_1) & -s(\phi_2 - \phi_1) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -s(\phi_2 - \phi_1) \\ c(\phi_2 - \phi_1) \end{bmatrix}$$

$$R^T s_1'^P = B_{12} s_2'^P = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then,

$$\phi^{(3)}(q) = \begin{bmatrix} -(x_2 - x_1)s\phi_1 + (y_2 - y_1)c\phi_1 \\ s(\phi_2 - \phi_1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



$$\phi^{(4)}(p, t) \rightarrow \text{use 3.5.12.} \quad c=1 \quad j=2$$

$$v_1' = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad s_2'^P = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad s_1'^P = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\phi^{(4)}(q, t) = v_1'^T A_1^T (r_2 - r_1) + v_1'^T \cancel{B_{12}} s_2'^P - v_1'^T \cancel{R^T} s_1'^P - c(t) = 0$$

Then

$$\phi^{\square}(p, t) = [x_2 - x_1 \quad y_2 - y_1] \cdot \begin{matrix} \checkmark \\ \begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{matrix} = [x_2 - x_1 \quad y_2 - y_1] \begin{bmatrix} c\phi_1 \\ s\phi_1 \end{bmatrix} - c(t)$$

$$= (x_2 - x_1)c\phi_1 + (y_2 - y_1)s\phi_1 - c(t) = 0.$$

The overall set of constraints is :

$$\phi(q, t) = \begin{bmatrix} x_1 \\ y_1 \\ \phi_1 - 0.025t \\ -(x_2 - x_1)s\phi_1 + (y_2 - y_1)c\phi_1 \\ s(\phi_2 - \phi_1) \\ (x_2 - x_1)c\phi_1 + (y_2 - y_1)s\phi_1 - 0.1t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This represents a system of nonlinear equations. Given a time t_k , one can solve this system for q_k .

Next we have to compute the Jacobian ϕ_q . We can do this in two ways. First by looking at $\phi(q, t)$ above we can compute ϕ_q as follows:

$$\phi_q(q, t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ s\phi_1 & -c\phi_1 & -(x_2 - x_1)c\phi_1 - (y_2 - y_1)s\phi_1 & -s\phi_1 & c\phi_1 & 0 & 0 \\ 0 & 0 & -c(\phi_2 - \phi_1) & 0 & 0 & c(\phi_2 - \phi_1) & 0 \\ -c\phi_1 & -s\phi_1 & -(x_2 - x_1)s\phi_1 + (y_2 - y_1)c\phi_1 & c\phi_1 & s\phi_1 & 0 & 0 \end{bmatrix}$$

The second approach is to use the expression of the partial derivatives provided in the book.

$\phi_{q_1}^{(1)} \rightarrow$ use Eq. 3.2.5

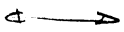
$$\phi_{q_1}^{(1)} = [I \quad B_{1, s_1^p}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \underbrace{\quad}_{\downarrow} & \underbrace{\quad}_{B_{1, s_1^p}} & \quad \end{bmatrix}$$

Where does $\phi_{q_1}^{(1)}$ go into the big ϕ_q ?

In $\phi(q, t)$, $\phi^{(1)}$ occupies rows 1 & 2.

$\phi_{q_1}^{(1)}$ is the partial with respect to entries 1, 2, and 3 of q .

Therefore $\phi_{q_1}^{(1)}$ goes in rows (1 & 2), columns (1 & 2 & 3).



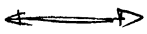
$\phi_{q_1}^{(2)} \rightarrow$ use Eq. 3.2.7

$$\phi_{q_1}^{(2)} = [0 \quad 0 \quad 1]$$

In $\phi(q, t)$, $\phi^{(2)}$ occupies row 3.

In q , q_1 occupies entries 1, 2, 3.

Then $\phi_{q_1}^{(2)}$ goes into ϕ_q in row 3, columns (1 & 2 & 3).



$\phi_{q_1}^{(3)} \rightarrow$ use Eq. 3.3.14.

$$\phi_{q_1}^{(3)} = \begin{bmatrix} -v_1'^T B_{11}^T & -v_1'^T A_{11}^T (R_2 - R_1) - v_1'^T A_{12} \begin{matrix} \nearrow \\ \downarrow \\ \nearrow \end{matrix} \\ 0 & -v_1'^T A_{12} v_2' \end{bmatrix}$$

$$-v_1'^T A_1^T (r_2 - r_1) = - \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} c\phi_1 & s\phi_1 \\ -s\phi_1 & c\phi_1 \end{bmatrix} (r_2 - r_1)$$

$$= \begin{bmatrix} -c\phi_1 & -s\phi_1 \end{bmatrix} \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} = -(x_2 - x_1)c\phi_1 - (y_2 - y_1)s\phi_1$$

$$-v_1'^T A_{12} v_2' = - \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} c(\phi_2 - \phi_1) & -s(\phi_2 - \phi_1) \\ s(\phi_2 - \phi_1) & c(\phi_2 - \phi_1) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = - \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} c(\phi_2 - \phi_1) \\ s(\phi_2 - \phi_1) \end{bmatrix}$$

$$= -c(\phi_2 - \phi_1)$$

Then,

$$\phi_{q_1}^{(3)} = \begin{bmatrix} s\phi_1 & -c\phi_1 & -(x_2 - x_1)c\phi_1 - (y_2 - y_1)s\phi_1 \\ 0 & 0 & -c(\phi_2 - \phi_1) \end{bmatrix}$$

In $\phi(q, t)$, $\phi^{(3)}$ occupies rows 4 & 5.

In q , q_1 occupies rows 1, 2, 3.

Then $\phi_{q_1}^{(3)}$ goes into ϕ_q in rows (4 & 5), columns (1 & 2 & 3).

Next we compute $\phi_{q_2}^{(3)}$. (see Eq. 3.3.14)

$$\phi_{q_2}^{(3)} = \begin{bmatrix} v_1'^T B_1^T & v_1'^T A_{12} v_2' \\ 0 & v_1'^T A_{12} v_2' \end{bmatrix} = \begin{bmatrix} -s\phi_1 & c\phi_1 & 0 \\ 0 & 0 & c(\phi_2 - \phi_1) \end{bmatrix}$$

In $\phi(q, t)$, $\phi^{(3)}$ occupies rows 4 & 5.

In q , q_2 occupies rows 4, 5, 6.

Then $\phi_{q_2}^{(3)}$ goes into ϕ_q in rows (4 & 5), columns (4 & 5 & 6).



To compute $\phi_{q_1}^{(A)}$ use Eq 3.5.13.

$$\phi_{q_1}^{(A)} = \begin{bmatrix} -v_1^T A^T & v_1^T B_1^T (\pi_2 - \pi_1) \end{bmatrix}$$

$$A v_1 = \begin{bmatrix} c\phi_1 & -s\phi_1 \\ s\phi_1 & c\phi_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c\phi_1 \\ s\phi_1 \end{bmatrix}$$

$$v_1^T B_1^T (\pi_2 - \pi_1) = \begin{bmatrix} -s\phi_1 & c\phi_1 \end{bmatrix} \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} = -(x_2 - x_1)s\phi_1 + (y_2 - y_1)c\phi_1$$

Then,

$$\phi_{q_1}^{(A)} = \begin{bmatrix} -c\phi_1 & -s\phi_1 & -(x_2 - x_1)s\phi_1 + (y_2 - y_1)c\phi_1 \end{bmatrix}$$

In $\phi(q, t)$, $\phi^{(A)}$ occupies row 6.

In q , q_1 occupies rows 1, 2, 3.

Then $\phi_{q_1}^{(A)}$ goes in ϕ_q in row 6, columns (1 & 2 & 3).

To compute $\phi_{q_2}^{(A)}$ use Eq. 3.5.13:

$$\phi_{q_2}^{(A)} = \begin{bmatrix} v_1^T A^T & 0 \end{bmatrix} = \begin{bmatrix} c\phi_1 & s\phi_1 & 0 \end{bmatrix}$$

In $\phi(q, t)$, $\phi^{(A)}$ occupies row 6.

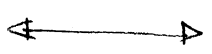
In q , q_2 occupies rows 4, 5, 6.

Then $\phi_{q_2}^{(A)}$ goes in ϕ_q in row 6, columns (4 & 5 & 6).

At this point we have all the information required to assemble ϕ_q (see next page).

$$\phi_q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ s\phi_1 & -c\phi_1 & -(x_2-x_1)c\phi_1 - (y_2-y_1)s\phi_1 & -s\phi_1 & c\phi_1 & 0 & 0 \\ 0 & 0 & -c(\phi_2 - \phi_1) & 0 & 0 & 0 & c(\phi_2 - \phi_1) \\ -c\phi_1 & -s\phi_1 & -(x_2-x_1)s\phi_1 + (y_2-y_1)c\phi_1 & c\phi_1 & s\phi_1 & 0 & 0 \end{bmatrix}$$

Note that this ϕ_q is identical to the one on page 4, which was determined by direct differentiation.

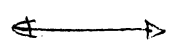


Next, to solve for velocity one needs $v = -\dot{\phi}_t$.

$$-\dot{\phi}_t = \begin{bmatrix} 0 \\ 0 \\ 0.025 \\ 0 \\ 0 \\ 0.1 \end{bmatrix}$$

The velocities are found after finding the position q_k as

$$\dot{q}_q \cdot \hat{q}_k = v$$



Finally, one has to solve for the accelerations, and for this we used $\ddot{\phi}$.

For $\phi^{(1)} \rightarrow$ use Eqs. at top of page 63.

$$x_1^{(1)p} = y_1^{(1)p} = 0 \Rightarrow \gamma = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For $\phi^{(2)} \rightarrow$ use 2.5.3

$$\gamma = \frac{\partial^2 C_3(t)}{\partial t^2} = 0.$$

For $\phi^{(3)} \rightarrow$ use Eq. on page 68 (note that a square bracket is missing in that equation)

$$\gamma^{t(1,2)} = \begin{bmatrix} \gamma_1^{t(1,2)} \\ 0 \end{bmatrix} = \gamma_1^{t(1,2)} \left[-\beta_{12} s_2'^p (\dot{\phi}_2 - \dot{\phi}_1)^2 + \beta_1^T (r_2 - r_1) \dot{\phi}_1 + 2A^T (r_2 - r_1) \dot{\phi}_1 \right]$$

$$s_2'^p = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\gamma_1^{t(1,2)} \beta_1^T = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -s\phi_1 & c\phi_1 \\ -c\phi_1 & -s\phi_1 \end{bmatrix} = \begin{bmatrix} -s\phi_1 & c\phi_1 \end{bmatrix}$$

$$\gamma_1^{t(1,2)} A^T = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} c\phi_1 & s\phi_1 \\ -s\phi_1 & c\phi_1 \end{bmatrix} = \begin{bmatrix} c\phi_1 & s\phi_1 \end{bmatrix}$$

$$\gamma^{t(1,2)} = \begin{bmatrix} \dot{\phi}_1^2 \begin{bmatrix} -s\phi_1 & c\phi_1 \end{bmatrix} \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} + 2 \begin{bmatrix} c\phi_1 & s\phi_1 \end{bmatrix} \begin{bmatrix} \dot{x}_2 - \dot{x}_1 \\ \dot{y}_2 - \dot{y}_1 \end{bmatrix} \dot{\phi}_1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \dot{\phi}_1^2 \left[-s\phi_1 (x_2 - x_1) + c\phi_1 (y_2 - y_1) + 2 \dot{\phi}_1 \left[c\phi_1 (\dot{x}_2 - \dot{x}_1) + s\phi_1 (\dot{y}_2 - \dot{y}_1) \right] \right] \\ 0 \end{bmatrix}$$

For $\phi^{(4)}$,

$$\delta^{(4)} = -2 \dot{\phi}_1 v_1'^T B_1^T (\dot{x}_2 - \dot{x}_1) + v_1'^T \dot{\phi}_1^2 A_1^T (x_2 - x_1) + \frac{\partial^2 c(t)}{\partial t^2}$$

$$= -2 \dot{\phi}_1 [-s\phi_1 \quad c\phi_1] \begin{bmatrix} \dot{x}_2 - \dot{x}_1 \\ \dot{y}_2 - \dot{y}_1 \end{bmatrix} + \dot{\phi}_1^2 [c\phi_1 \quad s\phi_1] \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$$

$$= -2 \dot{\phi}_1 [-s\phi_1 (\dot{x}_2 - \dot{x}_1) + c\phi_1 (\dot{y}_2 - \dot{y}_1)] + \dot{\phi}_1^2 [c\phi_1 (x_2 - x_1) + s\phi_1 (y_2 - y_1)]$$

Based on δ for $\phi^{(1)}$ through $\phi^{(4)}$, we assemble the overall δ :

$$\delta = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dot{\phi}_1^2 [-s\phi_1 (x_2 - x_1) + c\phi_1 (y_2 - y_1)] + 2\dot{\phi}_1 [c\phi_1 (\dot{x}_2 - \dot{x}_1) + s\phi_1 (\dot{y}_2 - \dot{y}_1)] \\ 0 \\ 2\dot{\phi}_1 [s\phi_1 (\dot{x}_2 - \dot{x}_1) - c\phi_1 (\dot{y}_2 - \dot{y}_1)] + \dot{\phi}_1^2 [c\phi_1 (x_2 - x_1) + s\phi_1 (y_2 - y_1)] \end{bmatrix}$$

We can verify the correctness of this result by taking two straight derivatives of our $\phi(q, t)$ (see page 4 of this document):

$$\dot{\phi}(q, t) = \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{\phi}_1 - 0.025 \\ -(\dot{x}_2 - \dot{x}_1) s\phi_1 - (x_2 - x_1) \dot{\phi}_1 c\phi_1 + (\dot{y}_2 - \dot{y}_1) c\phi_1 - (y_2 - y_1) \dot{\phi}_1 s\phi_1 \\ c(\phi_2 - \phi_1) (\dot{\phi}_2 - \dot{\phi}_1) \\ (\dot{x}_2 - \dot{x}_1) c\phi_1 - (x_2 - x_1) \dot{\phi}_1 s\phi_1 + (\dot{y}_2 - \dot{y}_1) s\phi_1 + (y_2 - y_1) \dot{\phi}_1 c\phi_1 - 0.1 \end{bmatrix}$$

x_1
 y_1
 $\dot{\phi}_1$

$$\phi(q, t) = \left[\begin{aligned} & -(\dot{x}_2 - \dot{x}_1) s\dot{\phi}_1 - 2(x_2 - x_1) \dot{\phi}_1 c\dot{\phi}_1 + (x_2 - x_1) \dot{\phi}_1^2 s\dot{\phi}_1 + (\ddot{y}_2 - \ddot{y}_1) c\dot{\phi}_1 - 2(\dot{y}_2 - \dot{y}_1) \dot{\phi}_1 s\dot{\phi}_1 - (y_2 - y_1) \dot{\phi}_1^2 s\dot{\phi}_1 \\ & \quad s(\dot{\phi}_2 - \dot{\phi}_1) (c\dot{\phi}_2 - c\dot{\phi}_1)^2 + c(\dot{\phi}_2 - \dot{\phi}_1) (\dot{\phi}_2^2 - \dot{\phi}_1^2) \\ & (\dot{x}_2 - \dot{x}_1) c\dot{\phi}_1 - 2(x_2 - x_1) \dot{\phi}_1 s\dot{\phi}_1 - (x_2 - x_1) \dot{\phi}_1^2 c\dot{\phi}_1 + (\ddot{y}_2 - \ddot{y}_1) s\dot{\phi}_1 + 2(\dot{y}_2 - \dot{y}_1) \dot{\phi}_1 c\dot{\phi}_1 + (y_2 - y_1) \dot{\phi}_1^2 c\dot{\phi}_1 - (y_2 - y_1) \dot{\phi}_1^2 s\dot{\phi}_1 \end{aligned} \right]$$

Then, separating δ out of $\phi(q, t)$ and ends up with

$$\delta = \left[\begin{aligned} & 2\dot{\phi}_1 [(x_2 - x_1) c\dot{\phi}_1 + (y_2 - y_1) s\dot{\phi}_1] + \dot{\phi}_1^2 [-(x_2 - x_1) s\dot{\phi}_1 + (y_2 - y_1) c\dot{\phi}_1] \\ & \quad 0 \\ & \quad 0 \\ & \quad 0 \\ & 2\dot{\phi}_1 [(\dot{x}_2 - \dot{x}_1) s\dot{\phi}_1 - (\dot{y}_2 - \dot{y}_1) c\dot{\phi}_1] + \dot{\phi}_1^2 [(x_2 - x_1) c\dot{\phi}_1 + (y_2 - y_1) s\dot{\phi}_1] \end{aligned} \right]$$

Note that, as expected, this is identical to δ derived at page 10 by using the δ formulas provided by the book.

Putting things in perspective, we derived all the information needed to carry out position, velocity and acceleration analysis.

We used the formulas provided in the book to find the partial derivatives required to assemble ϕ_q , we used appropriate formulas to come up with v , and finally to produce the RHS of the acceleration equation.

Note that we had to assemble pieces of information provided by the formulas based on the global index associated with each constraint and each generalized coordinate. In this context, $\phi^{(1)}$ had global index 1 & 2, $\phi^{(2)}$ global index 3, $\phi^{(3)}$ had global index 4 & 5, and $\phi^{(4)}$ had global index 6. By the same token, x_1, y_1, ϕ_1 had global index 1, 2, and 3, while the generalized coordinates associated with body 2 had global index in the overall generalized coordinate vector 4, 5, and 6.

These global indices are important when assembling ϕ, v, \dot{v} , and ϕ_q .

