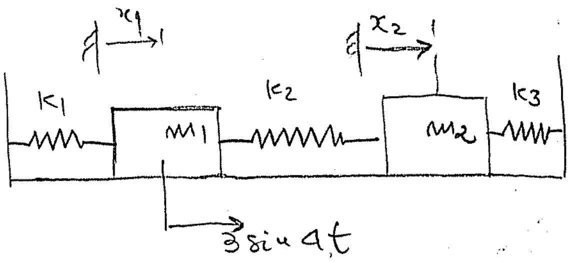
Example – Undamped Forced Vibration 1/2
~ Modal Analysis Approach ~



Initial conditions.

$$\{x_0\} = \begin{Bmatrix} 3 \\ 0 \end{Bmatrix} \qquad \{\dot{x}_0\} = \begin{Bmatrix} 0 \\ 9 \end{Bmatrix}$$

EOM:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 3\sin 9t \\ 0 \end{Bmatrix}$$

$m_1 = 1$

$m_2 = 2$

$k_1 = 9$

$k_2 = k_3 = 18$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 27 & -18 \\ -18 & 36 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 3\sin 9t \\ 0 \end{Bmatrix}$$

For this problem, $[u] = \begin{bmatrix} 1 & 1 \\ 1 & -0.5 \end{bmatrix} \Rightarrow [u]^{-1} = \begin{bmatrix} 1/3 & 2/3 \\ 2/3 & -2/3 \end{bmatrix}$

$$\{f(t)\} = \begin{Bmatrix} 3\sin 9t \\ 0 \end{Bmatrix} \Rightarrow [u]^T\{f(t)\} \equiv F(t) = \begin{bmatrix} 1 & 1 \\ 1 & -0.5 \end{bmatrix} \begin{Bmatrix} 3\sin 9t \\ 0 \end{Bmatrix}$$

$$\Rightarrow \{F(t)\} = \begin{Bmatrix} 3\sin 9t \\ 3\sin 9t \end{Bmatrix}$$

Then, the differential equation satisfied by the principal coordinates are:

$$\underbrace{[u]^T[m][u]}_{[M]}\ddot{q} + \underbrace{[u^T][k][u]}_{[K]}q = \underbrace{[u]^T\{f(t)\}}_{\{F(t)\}}.$$

Recall that $[u]$ diagonalizes both $[m]$ and $[k]$, see class discussion on modal analysis.

Note that both [M] and [K] are diagonal:

$$[M] = \begin{bmatrix} 3 & 0 \\ 0 & 1.5 \end{bmatrix} \qquad \{F(t)\} = \begin{Bmatrix} 3 \\ 3 \end{Bmatrix} \cdot \sin 4t$$

$$[K] = \begin{bmatrix} 27 & 0 \\ 0 & 54 \end{bmatrix}$$

The initial conditions for our problem are:

$$\{q_0\} = [u]^{-1}\{x_0\} \qquad \{\dot{q}_0\} = [u]^{-1} \cdot \{\dot{x}_0\}$$

Since $\{x_0\} = \begin{Bmatrix} 3 \\ 0 \end{Bmatrix}$ and $\{\dot{x}_0\} = \begin{Bmatrix} 0 \\ 9 \end{Bmatrix}$,

$$\{q_0\} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \quad \text{and} \quad \{\dot{q}_0\} = \begin{Bmatrix} 6 \\ -6 \end{Bmatrix}$$

Then, the decoupled EOMs assume the form:

$$\begin{cases} 3\ddot{q}_1 + 27q = 3\sin 4t & q_1(0) = 1 \quad \dot{q}_1(0) = 6 \\ 1.5\ddot{q}_2 + 54q = 3\sin 4t & q_2(0) = 2 \quad \dot{q}_2(0) = -6 \end{cases}$$

At this point you are in a position to solve each of these IVPs. Once you have $q_1(t)$ and $q_2(t)$, you get $x(t)$ as

$$\{x(t)\} = [u] \cdot \{q\} = \begin{bmatrix} 1 & 1 \\ 1 & -0.5 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} q_1(t) + q_2(t) \\ q_1(t) - 0.5\,q_2(t) \end{bmatrix}$$