EXAMPLE 2-7

Let us use the concept of $T_{max} = U_{max}$ to determine the undamped natural circular frequency ω_n of the system shown in Fig. 2-23 which consists of a weight W attached to a slender rod. The rod has a length l, a mass per unit length γ, and a stiffness factor $k = AE/l$ (see Table 2-1 on page 79).

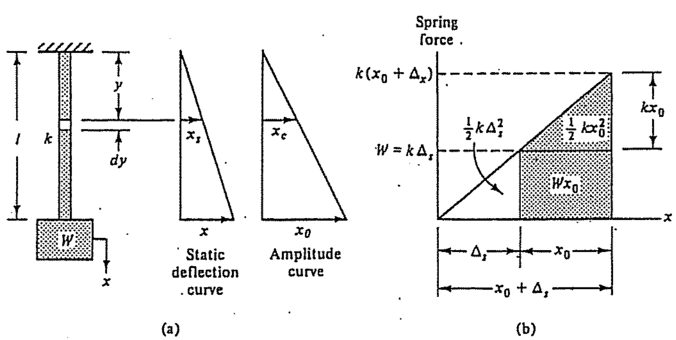

Figure 2-23 Spring-and-mass system with a slender rod as spring.

Solution. Our first step is to select a displacement function to use in determining the kinetic and potential energy of the system. We know from elementary mechanics of materials that the static displacement x_s of any cross section of the rod is proportional to its distance y from the fixed end. Thus, the static-deflection curve is as shown in Fig. 2-23a. If we now make the reasonable assumption that the *amplitude curve* of the rod has the same *shape* as the static-deflection curve as indicated in Fig. 2-23a, we can use as our shape function

$$x_c = \frac{x_0 y}{l}$$

in which x_c is the *amplitude* of any cross section of the rod and x_0 is the *amplitude* of the lower end of the rod and the weight W, as shown in Fig. 2-23a.

Since the natural circular frequency ω_n of each cross section of the rod is the same, the *maximum* velocity of any cross section along the rod is simply

$$x_c\omega_n = \frac{x_0 y}{l}\omega_n$$

The maximum kinetic energy of a differential element of length dy of the rod is

$$dT = \frac{1}{2}\gamma\left(\frac{x_0 y \omega_n}{l}\right)^2 dy$$

in which $\gamma\, dy$ is the mass of the differential element.

The maximum kinetic energy *of the system* is

$$T_{max} = \frac{1}{2}\frac{W}{g}(x_0\omega_n)^2 + \frac{1}{2}\gamma\int_0^l \left(\frac{x_0 y}{l}\right)^2 \omega_n^2\, dy$$

Integrating and simplifying,

$$T_{max} = \left[\frac{M}{2} + \frac{m_s}{2(3)}\right]x_0^2\omega_n^2 \tag{2-53}$$

where $M = W/g$ = mass of weight W
$m_s = \gamma l$ = mass of rod

Referring to the spring-force diagram in Fig. 2-23b we see that the strain energy in the static-equilibrium position is $k\Delta_s^2/2$. Therefore, the *maximum change* in the strain energy U_e for a positive amplitude x_0 measured from the static-equilibrium position is the shaded area in the figure

$$U_e = \tfrac{1}{2}kx_0^2 + Wx_0$$

The accompanying *maximum change* in the potential energy of position U_g as the weight W moves x_0 *below* the static-equilibrium position is

$$U_g = -Wx_0$$

The maximum change in the potential energy U of the system from its potential energy in the static-equilibrium position is thus

$$U_{max} = U_e + U_g = \tfrac{1}{2}kx_0^2 \tag{2-54}$$

Since $T_{max} = U_{max}$, we set Eq. 2-53 equal to Eq. 2-54 to find that

$$\omega_n = \sqrt{\frac{k}{M + m_s/3}}$$

or

$$\omega_n = \sqrt{\frac{k}{M(1 + \alpha/3)}} \tag{2-55}$$

in which the mass ratio $(m_s/M) = \alpha$.

It should be apparent from Eq. 2-55 that if the mass m_s of the rod is small in comparison with the mass M of the rigid body, it has little effect on the natural circular frequency ω_n of the system. The mass term $(M + m_s/3)$ is frequently referred to as the *effective mass* M_e of a system. The approximate solution given by Eq. 2-55 is within 1 percent of the exact solution of the wave equation obtained in Sec. 8-3 when $m_s/M = 1.0$. However, as m_s/M increases above this value, the approximate solution begins to deviate somewhat from the exact solution. When $m_s/M = 5$, for example, the error is on the order of 4 percent. ■■