Two small gears are attached to a slender shaft that is supported by the self-aligning bearings A and B as shown in part a of the accompanying figure. The engineer designing the system would like to determine the natural frequencies of the system experimentally but does not have the equipment necessary to do so. Therefore, rather than obtaining vibration measurements, he applies a 20-lb static load at gear 1 and measures the static deflections of the shaft at the location of the center of each gear as shown in part b of the figure. He then applies a 20-lb force statically at gear 2 and again measures the static deflections of the shaft at the location of the center of each gear as shown in part c of the figure. The natural frequencies and mode shapes are then calculated by using these static-load test data with the assumption that the mass of the shaft is negligible. Determine the frequencies and mode shapes calculated by the engineer.

Determine the influence coefficients:

\[
\begin{align*}
a_{11} &= 0.1, \quad a_{20} = 0.005 \text{ in./lb} \\
a_{12} &= -0.04, \quad a_{21} = 0.002 \text{ in./lb} \\
a_{22} &= 0.06, \quad a_{30} = 0.003 \text{ in./lb}
\end{align*}
\]

Determine the mass matrix:

\[
\begin{align*}
M_1 &= \frac{1}{36} = 0.01036 \text{ lb-in}^2 \\
M_2 &= \frac{3}{36} = 0.00777 \text{ lb-in}^2
\end{align*}
\]

\[
|D| = \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix}
\]

Let \( \lambda = 0 \), then

\[
(a_{11} - \lambda)(a_{22} - \lambda) - a_{21} a_{12} m_1 m_2 = 0
\]

\[
\lambda^2 - (a_{11} + a_{22}) \lambda + a_{11} a_{22} - a_{21} a_{12} m_1 m_2 = 0
\]

Substituting the values

\[
\lambda^2 - 7.511 \times 10^5 \lambda + 4.8547 \times 10^{10} = 0
\]
\[
\lambda_1 = \frac{7.51 \times 10^{-5}}{2} \sqrt{\frac{(7.54 \times 10^{-5})^2}{4 \times (8.8547 \times 10^{-10})}}
\]

\[
\lambda_2 = 6.04 \times 10^{-5}
\]

\[
\omega = \frac{1}{\sqrt{\lambda_1}} = 128.6 \text{ rad/s}
\]

\[
f = \frac{\omega}{2\pi} = 20.47 \text{ Hz}
\]

\[
\lambda_2 = 1.46 \times 10^{-5}
\]

\[
\omega_2 = \frac{1}{\sqrt{\lambda_2}} = \frac{261.32 \text{ rad/s}}{\sqrt{5}}
\]

\[
f_2 = \frac{\omega_2}{2\pi} = 44.51 \text{ Hz}
\]

From these data, we have:

\[
\begin{align*}
\sum [x \bar{m}] &= \frac{1}{\omega^2} \sum [x \bar{m}] = \frac{8}{5} 0^2 \\
\end{align*}
\]

Then:

\[
\begin{bmatrix}
(a_{11} m_1 - \frac{1}{\omega^2}) & a_{12} m_2 \\
a_{21} m_1 & (a_{22} m_2 - \frac{1}{\omega^2})
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
(5.18 \times 10^{-5} - \frac{1}{\omega^2}) & (-1.55 \times 10^{-5}) \\
(-1.55 \times 10^{-5}) & (2.35 \times 10^{-5} - \frac{1}{\omega^2})
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

\[
\begin{align*}
X_1 &= \frac{1.55 \times 10^{-5}}{\omega^2} \\
X_2 &= \frac{5.18 \times 10^{-5} - \frac{1}{\omega^2}}{\omega^2}
\end{align*}
\]

Model 1:

\[
\frac{X_1}{X_2} = \frac{-1.9}{1.9} \implies \frac{X_1}{X_2} = -0.529
\]

Model 2:

\[
\frac{X_1}{X_2} = \frac{0.417}{1.9} \implies \frac{X_1}{X_2} = 0.217
\]

\[
\omega = 261.2 \text{ rad/s}
\]

\[
\frac{X_1}{X_2} = 0.417 \quad \Rightarrow \quad \frac{X_1}{X_2} = 0.40
\]