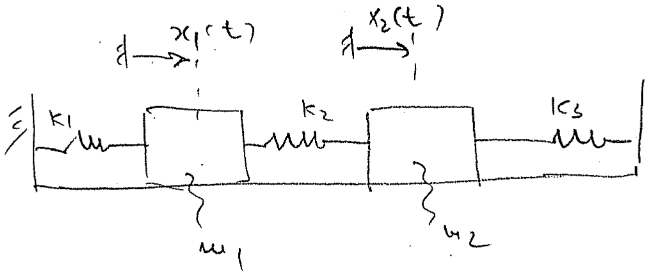
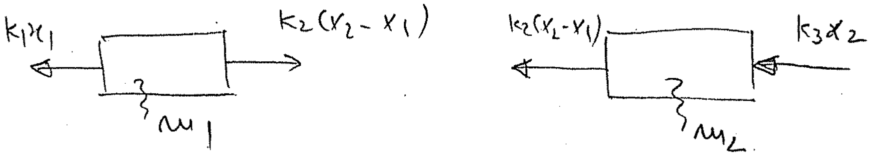


$m_1 = 1 \qquad m_2 = 2$

$k_1 = 9$

$k_2 = k_3 = 18$

$$\text{IC}\begin{cases} x_1(0)=3 & \dot{x}_1(0)=0 \\ x_2(0)=0 & \dot{x}_2(0)=9 \end{cases}$$



Newton's Second Law:

$$\begin{cases} m_1\ddot{x}_1 = k_2(x_2-x_1) - k_1x_1 \\ m_2\ddot{x}_2 = -k_2(x_2-x_1) - k_3x_2 \end{cases}$$

$$\begin{cases} m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 = 0 \\ m_2\ddot{x}_2 - k_2x_1 + (k_2+k_3)x_2 = 0 \end{cases} \Rightarrow \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0$$

Based on the course notes:

$k_{11} = k_1 + k_2 = 27$

$k_{12} = -k_2 = -18$

$k_{22} = k_2 + k_3 = 36.$

Then, natural frequencies are the solution of

$$1\cdot 2\omega^4 - (1\cdot 36 + 2\cdot 27)\,\omega^2 + 27\cdot 36 - (18)^2 = 0$$

$$2\omega^4 - 90\omega^2 + 648 = 0 \Rightarrow \begin{cases} \omega_{n(1)}^2 = 9 \Rightarrow \omega_{n(1)} = 3 \\ \omega_{n(2)}^2 = 36 \Rightarrow \omega_{n(2)} = 6 \end{cases}$$

$$r_{(1)} = -\frac{-18}{36 - 2\cdot 3^2} = \frac{18}{36-18} = 1$$

$$r_{(2)} = -\frac{-18}{36-2\cdot 6^2} = \frac{18}{36-72} = -\frac{1}{2}$$

Then $\{u\}_{(1)} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$ $\quad \{u\}_{(2)} = \begin{Bmatrix} 1 \\ -0.5 \end{Bmatrix}$

Therefore, the solution assumes the form

$$\{x(t)\} = \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = A^{(1)}\{u\}_{(1)}\cos(\omega_{n(1)}t - \varphi_{(1)}) + A^{(2)}\{u\}_{(2)}\cos(\omega_{n(2)}t - \varphi_{(2)})$$

Find $A^{(1)}$, $A^{(2)}$, $\varphi_{(1)}$, $\varphi_{(2)}$ based on ICs:

$\{x(0)\} = \begin{Bmatrix} 3 \\ 0 \end{Bmatrix}$ $\quad \{\dot{x}(0)\} = \begin{Bmatrix} 0 \\ 9 \end{Bmatrix}$

$$\{x(0)\} = A^{(1)}\{u\}_{(1)}\cos\varphi_{(1)} + A^{(2)}\{u\}_{(2)}\cos\varphi_{(2)} = \begin{Bmatrix} 3 \\ 0 \end{Bmatrix}$$

$$\{\dot{x}(t)\} = -\omega_{n(1)}A^{(1)}\{u\}_{(1)}\sin(\omega_{n(1)}t - \varphi_{(1)}) - \omega_{n(2)}A^{(2)}\{u\}_{(2)}\sin(\omega_{n(2)}t - \varphi_{(2)})$$

$$\dot{x}(0) = \omega_{n(1)}\cdot A^{(1)}\{u\}_{(1)}\sin\varphi_{(1)} + \omega_{n(2)}A^{(2)}\{u\}_{(2)}\sin\varphi_{(2)} = \begin{Bmatrix} 0 \\ 9 \end{Bmatrix}$$

Equivalently:

$$\begin{cases} A^{(1)} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \cos \varphi_{(1)} + A^{(2)} \begin{Bmatrix} 1 \\ -0.5 \end{Bmatrix} \cos \varphi_{(2)} = \begin{Bmatrix} 3 \\ 0 \end{Bmatrix} \\ 3 A^{(1)} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \sin \varphi_{(1)} + 6 A^{(2)} \begin{Bmatrix} 1 \\ -0.5 \end{Bmatrix} \sin \varphi_{(2)} = \begin{Bmatrix} 0 \\ 9 \end{Bmatrix}. \end{cases}$$

A system with 4 equations and four unknowns.

solution of the system is:

$A^{(1)} = \sqrt{5}$ $\quad \varphi_{(1)} = 1.10715$ $\quad A^{(2)} = \sqrt{5}$ $\quad \varphi_{(2)} = -0.46365.$

Then,

$$\boxed{\{x(t)\} = \sqrt{5} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \cos(3t - 1.10715) + \sqrt{5} \begin{Bmatrix} 1 \\ -0.5 \end{Bmatrix} \cos(6t + 0.46365)}$$