\[ x(t) = x_0 \cos \omega_n t + \frac{x_0}{\omega_n} \sin \omega_n t + \frac{F_0}{k_c} \left[ \frac{\omega_n}{\omega_n^2 - \omega_n^2} \cos \omega_n t_0 - \omega_n \sin \omega_n t_0 \right] \\
+ \frac{\omega_n}{\omega_n^2 - \omega_n^2} \sin \omega_n (t - t_0) + \omega_n \cos \omega_n (t - t_0) - \frac{\omega_n}{\omega_n^2 - \omega_n^2} \sin \omega_n t_0 \]

The simply supported beam shown in part a of the accompanying figure is modeled as a single-degree-of-freedom system in which the distributed mass of the beam is lumped as a single mass \( m \) that is equal to one-half the total mass of the beam and that is located at the center of the beam as shown in part b of the figure. A pneumatic loading device is used to subject the beam to a simulated blast-type load

\[ F(t) = F_0 e^{-kt} \]

which decreases exponentially with time as shown in part c of the figure. Show that the response of the simplified model of the beam at \( x = l/2 \) is

\[ y = \frac{F_0 l^3}{48EI[1 + (b/\omega_n)^2]} \left( e^{-\omega_n t} \cos \omega_n t + \frac{b}{\omega_n} \sin \omega_n t \right) \]

where
- \( E1 \) = stiffness factor of beam
- \( l \) = length of beam
- \( \omega_n = \sqrt{96EI/m_l^3} \)

The total mass of the beam is \( m_b = 2 \text{ m} \). Assume \( \ddot{y}_b = 0 \) and \( \dot{y}_b = 0 \).


\[ F(t) \]
\[ m = \frac{m_0}{1} \]
\[ \omega_n = \sqrt{\frac{k}{m}} \]
\[ \omega_k = \frac{k}{m} \]
\[ m = m_0 \]
\[ \text{Initial Conditions:} \]
\[ y_0 = \dot{y}_0 = 0 \]
\[ y(0) = A + \frac{F_0}{m(b^2 + \omega_n^2)} = 0 \quad \text{or} \quad A = -\frac{F_0}{m(b^2 + \omega_n^2)} \]
\[ \text{Differential Eqns of Motion:} \]
\[ \ddot{y} + \omega_n^2 \dot{y} = \frac{F_0}{m} e^{-bt} \]
\[ y(t) = -A \omega_n \sin \omega_n t + B \omega_n \cos \omega_n t - \frac{b F_0}{m(b^2 + \omega_n^2)} e^{-bt} \]
\[ y(0) = B \omega_n - \frac{b F_0}{m(b^2 + \omega_n^2)} = 0 \quad \text{or} \quad B = \frac{b F_0}{m \omega_n (b^2 + \omega_n^2)} \]

\[ \text{Solution 1 - Direct Integration of Eqn of Motion:} \]
\[ y_p = A \cos \omega_n t + B \sin \omega_n t \]
\[ \ddot{y}_p = -b C e^{-bt} \]
\[ \dot{y}_p = b^2 C e^{-bt} \]
\[ y(t) = \frac{F_0}{m(b^2 + \omega_n^2)} \left[-\cos \omega_n t + \frac{b}{\omega_n} \sin \omega_n t + e^{-bt}\right] \]

\[ \text{Substituting } \dot{y}_p \text{ into ODE:} \]
\[ F_0 = m \omega_n^2 \dot{y}_p \]
\[ \ddot{y}_p = \frac{F_0}{m} e^{-bt} \]
\[ \ddot{y}_p - \omega_n^2 \dot{y}_p = \frac{F_0}{m} e^{-bt} \]
\[ b^2 C + \omega_n^2 C = \frac{F_0}{m} \]
\[ C = \frac{F_0}{m (b^2 + \omega_n^2)} \]
\[ \text{Solution 2 - Duhamel's Integral} \]
\[ y(t) = \int_0^t F(t') g(t-t') \, dt' = \int_0^t F(t') g(t) \, dt \]
\[ y(t) = \int_0^t F(t') e^{-bt'} \, dt' \]
\[ g(t) = \frac{1}{m \omega_n} \sin \omega_n t \quad F(t) = F_0 e^{-bt} \]
\[ y(t) = \int_{t_0}^{t} e^{-bt} \left( \frac{m}{I} \right)^{\frac{1}{2}} \sin \left( \frac{\omega_n}{2} \left( t - t_0 \right) \right) \, dt = \int_{t_0}^{t} e^{-bt} \left( \frac{m}{I} \right)^{\frac{1}{2}} \sin \left( \frac{\omega_n}{2} \left( t - t_0 \right) \right) \, dt \]

\[ y(t) = \frac{F_o}{m \omega_n} \int_{t_0}^{t} e^{-bt} \left[ -\cos \omega_n (t - t_0) + \frac{b}{\omega_n} \sin \omega_n (t - t_0) \right] \, dt \]

From CEC:
\[ \int e^{-at} \sin bt \, dt = \frac{e^{-at}}{a^2 + b^2} \left[ a \sin bt - b \cos bt \right] \]

Then,
\[ y(t) = \frac{F_o}{m \omega_n} \int_{t_0}^{t} e^{-bt} \left[ -\cos \omega_n (t - t_0) + \frac{b}{\omega_n} \sin \omega_n (t - t_0) \right] \, dt \]

\[ y(t) = \frac{F_o}{m \omega_n (b^2 + \omega_n^2)} \left[ -\cos \omega_n (t - t_0) + \frac{b}{\omega_n} \sin \omega_n (t - t_0) - e^{-bt} \right] \]

\[ y(t) = \frac{F_o}{m (b^2 + \omega_n^2)} \left[ -e^{-bt} - \cos \omega_n (t - t_0) + \frac{b}{\omega_n} \sin \omega_n (t - t_0) \right] \]

\[ y(t) = \frac{F_o}{m (b^2 + \omega_n^2)} \int_{t_0}^{t} e^{-bt} \left[ -\cos \omega_n (t - t_0) + \frac{b}{\omega_n} \sin \omega_n (t - t_0) \right] \, dt \]

\[ y(t) = \frac{F_o l^2}{48EI} \left[ 1 + \frac{b^2}{2 \omega_n^2} \right] \left[ -e^{-bt} - \cos \omega_n (t - t_0) + \frac{b}{\omega_n} \sin \omega_n (t - t_0) \right] \]