Determining the Total Response

I.) Direct Integration

\[ x(t) = x_h(t) + x_p(t) \]

a.) Assume a solution for \( x_p(t) \)

b.) Substitute \( x_p(t) \) back into the ODE to determine all constants

c.) For \( x_h(t) \) assume:

\[ x_h(t) = A\cos\omega_h t + B\sin\omega_h t \quad \text{(undamped)} \]

\[ x_h(t) = e^{-\xi \omega_d t} (A\cos\omega_d t + B\sin\omega_d t) \quad \text{(damped)} \]

d.) Solve for \( A + B \) using the complete response \( x(t) \)

\[ x(t) = x_h(t) + x_p(t) \]

II.) Duhem's Integral

\[ x(t) = x_h(t) + x_{Duh}(t) \]

(Undamped)

\[ x(t) = x_0 \cos\omega_h t + \frac{x_0}{\omega_h} \sin\omega_h t + x_{Duh}(t) \]

(Damped)

\[ x(t) = e^{-\xi \omega_d t} \left[ x_0 \cos\omega_d t + \left( \frac{x_0 + \xi \omega_d x_0}{\omega_d^2} \right) \sin\omega_d t \right] \]

+ \( x_{Duh}(t) \)

The frame, anvil and the base of the forging hammer, shown in Fig. (a), have a total mass of \( m \). The support elastic pad has a stiffness of \( k \). If the force applied by the hammer is given by Fig. (b), find the response of the anvil. Assume initial conditions of \( x_0 \) and \( \dot{x}_0 \).
Range I: \( 0 \leq t \leq t_0 \)

\[
x(t) = x_h(t) + x_{bh}(t)
\]

\[
x_h(t) = \int_0^t \frac{F_0}{k} \sin \omega_n (t-\tau) d\tau
\]

\[
x_{bh}(t) = \frac{F_0}{k} \left( \frac{t}{t_0} - \sin \frac{\omega_n t}{t_0} \right)
\]

\[
x(t) = x_h \cos \omega_n t + \frac{x_0}{\omega_n} \sin \omega_n t + \frac{F_0}{k} \left( \frac{t}{t_0} - \sin \frac{\omega_n t}{t_0} \right)
\]

\[
X_{bh}(t) = \frac{F_0 \omega_n}{t_0 k} \left[ \sin \frac{\omega_n (t-t_0)}{\omega_n^2} + \frac{t_0}{\omega_n} \cos \frac{\omega_n (t-t_0)}{\omega_n^2} - \frac{\sin \omega_n t}{\omega_n^2} \right] - \frac{F_0 \omega_n}{k} \left[ \cos \omega_n t - \cos \omega_n t_0 \right]
\]

\[
X_{bh}(t) = \frac{F_0}{k} \left[ \sin \frac{\omega_n (t-t_0)}{\omega_n t_0} + \cos \omega_n (t-t_0) - \frac{\sin \omega_n t}{\omega_n^2} \right] + \frac{F_0}{k} \left[ \omega_n t_0 \omega_n t - \omega_n \omega_n t \right]
\]

Range II: \( t_0 \leq t \leq 5t_0 \)

\[
x(t) = x_h(t) + x_{bh}(t)
\]

\[
x_{bh}(t) = \int_0^{t_0} \frac{F_0}{k} \sin \omega_n (t-\tau) d\tau + \int_{t_0}^t \frac{F_0}{k} \sin \omega_n \tau d\tau
\]

\[
x_{bh}(t) = \frac{F_0 \omega_n}{t_0 k} \int_0^{t_0} \sin \omega_n (t-\tau) d\tau + \frac{F_0 \omega_n}{k} \int_{t_0}^t \sin \omega_n \tau d\tau
\]

\[
x_{bh}(t) = \frac{F_0 \omega_n}{t_0 k} \left[ \sin \frac{\omega_n (t-t_0)}{\omega_n^2} + \frac{t_0}{\omega_n} \cos \frac{\omega_n (t-t_0)}{\omega_n^2} - \frac{\sin \omega_n t}{\omega_n^2} \right]_{t_0}^t - \frac{F_0 \omega_n}{k} \left[ \cos \omega_n t - \cos \omega_n t_0 \right]
\]

Range III: \( t \geq 5t_0 \)

\[
x(t) = x_h(t) + x_{bh}(t)
\]

\[
x_{bh}(t) = \int_0^{5t_0} \frac{F_0}{k} \sin \omega_n (t-\tau) d\tau + \int_{5t_0}^t \frac{F_0}{k} \sin \omega_n \tau d\tau
\]

\[
x_{bh}(t) = \frac{F_0 \omega_n}{t_0 k} \int_0^{5t_0} \sin \omega_n (t-\tau) d\tau + \frac{F_0 \omega_n}{k} \int_{5t_0}^t \sin \omega_n \tau d\tau
\]

\[
x_{bh}(t) = \frac{F_0}{k} \left[ \sin \frac{\omega_n (t-t_0)}{\omega_n t_0} + \cos \omega_n (t-t_0) - \frac{\sin \omega_n t}{\omega_n^2} \right]_{t_0}^t + \frac{F_0}{k} \left[ \omega_n t_0 \omega_n t - \omega_n \omega_n t \right]
Example 2

The simply supported beam shown in part a of the accompanying figure is modeled as a single-degree-of-freedom system in which the distributed mass of the beam is lumped as a single mass \( m \) that is equal to one-half the total mass of the beam and that is located at the center of the beam as shown in part b of the figure. A pneumatic loading device is used to subject the beam to a simulated blast-type load

\[
F(t) = F_0 e^{-bt}
\]

which decreases exponentially with time as shown in part c of the figure. Show that the response of the simplified model of the beam at \( x = l/2 \) is

\[
y = \frac{F_0 l^3}{48EI[1 + (b/\omega_n)^2]} \left( e^{-bt} - \cos \omega_n t + \frac{b}{\omega_n} \sin \omega_n t \right)
\]

where \( EI \) = stiffness factor of beam

\( l \) = length of beam

\( \omega_n = \sqrt{96EI/mL^2} \)

The total mass of the beam is \( m_b = 2 \) m. Assume \( \dot{u}_b = 0 \) and \( \ddot{u}_b = 0 \).